ON THE ALGEBRA OF IDEALS AND MODULES IN OPERATOR SPACES

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TITLE PAGE

DECLARATION

This research thesis is my original work prepared with no other than the indicated sources and support and has not been presented elsewhere for award of degree or any other award.

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CERTIFICATION

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DEDICATION

I dedicate this work to God, my late father Samson Wanjala Makila and my immediate family.

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This has been the longest journey in my academic pursuit. Therefore this is indeed a deserving achievement. The road to this ultimate achievement was full of many challenges, setbacks and good times too. I would not have reached here were it not for the many people in my life playing various significant roles. First thanks be to God for the grace and overwhelming favour. God is the anchor of my life on whose word I drew great inspiration when doubling as an academician and a church pastor. He helped me to recognize the value of forging along anyway despite the rough terrain and to keep my poise and enable me to navigate adeptly through moments of uncertainty. To Him be the glory. My supervisors: Dr. Ojiema Michael Onyango and Dr. Simiyu Achiles Nyongesa for guidance in choice of topic, supervision and constant encouragement and patience. You have been quite patient and understanding to help me reach here. Dr. Ojiema Michael Onyango, I am very much indebted to your insight and unmatched sacrifice. Only God can pay you back. My Mother Ann Namalwa who believes so much in me and would constantly remind me that with God I was set to make it. My wife and true companion Dr Leunita Makutsa Makila who used all means at her disposal to help me balance out to have this work completed. My children Derek, Laventine, and Tim for understanding and constantly reminding me that I was next on the graduating list in the family. Many special old friends , fellow mathematicians represented by Dr. Marani Vincent with whom we held a lot of discussions. I appreciate you all.

ABSTRACT

The Theory of Operator Ideals and Frechet Modules are important in the study of locally convex spaces, Rings and Algebras. Locally convex spaces are examples of topological vector spaces which generalize normed spaces, so they are Frechet in nature. The original idea in this line was meant to get the interplay between the operator spaces and their subspaces which exhibit the ideal properties from the algebraic point of view. The well known ideals have got certain restrictions on the projections in the spaces, their duals and annihilators. The $M-$embedments, one- sided structures, multipliers and related theories of r, $l-$ideals were developed with a hope to enrich the non-commutative attributes and a generalization of ideal structures to specified operator spaces and the clarity of Algebras of operators on Banach spaces, as well as homomorphisms thereof. Despite the fact that studies concerning the Algebra of Ideals and Modules in operator spaces with applications is still active, their general classification and extension remain unsettled. Therefore, the main objective of this study was to characterize the Algebra of Ideals and Modules in certain Operator Spaces. To achieve the objective, we determined the classification of ideals in the set of operators in Banach spaces, characterized the spaces of ideal operators and ideal extensions to Frechet spaces, extended the approximation properties of the ideals through the Integral and Nuclear Operators and determined the algebra of Banach Modules and Functors over the Frechet Spaces. The study employed the methods independently proposed by Godfrey,Kalton, and Saphar, and Sonia to characterize the operator ideals. The hypocontinuity criterion of the Module functors in the Frechet spaces followed the methods proposed by Rieffel. The results demonstrate the existence of classes of closed operator ideals depicting the boundedness in view of Radon-Nikodym properties. Additionally, the findings give the characteristics of ideals through the Hahn-Banach extension operators as well as the necessary and sufficient conditions for the existence of $u, h-$ideals and their variants in generalized Banach spaces. Finally, the relationship between categorical products and co-products of kernels from one module to the other, flatness of the Module Tensors, and the fact that given an interactive system of modules in a bounded Banach algebra, all canonical morphisms from the module to the collections of its isometric immersions is an isomorphism have been determined. Further characteristics of Frechet modules including; strong factorization properties over the functors, continuity and hypo-continuity of the multiplication have been determined. The findings of this research are significant because of the topological interplay between ideals and modules which allows the hereditary properties of ideals to be used to study modules and opens an area of interest in Algebra. Furthermore, the results display the interplay between Algebra and Analysis, hence contributing to the body of knowledge in both disciplines.

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INDEX OF NOTATIONS AND SYMBOLS

$\mathbf{X}, \mathbf{Y}$	:	Arbitrary Banach spaces
$\mathbf{K(X,Y)}$	:	Space of compact linear operators from X to Y
$\mathbf{L(X,Y)}$	:	Space of all bounded linear operators from X to Y
$\mathbf{L}(\mathbf{X}, \mathbf{Y})^*$	:	Dual space of $L(X, Y)$
$\mathbf{F(X,Y)}$	:	Set of finite rank operators from Banach space X to Y
$\overline{\mathbf{F}(\mathbf{X}, \mathbf{Y})}$	:	Closure of space of finite rank operators from Banach space X to Y
$\mathbf{B_X}$	:	A closed unit ball of X
$\mathbf{B}(\mathbf{F}, \mathbf{E})$	:	Set of Hahn-Banach extension operators $\Phi : F^* \longrightarrow E^*$
$\hat{\mathbf{X}}$	:	Equivalent renorming of the Banach space X
$\mathbf{l_1}$	:	Space of 1-summable sequence
UKAP	:	Unconditional compact approximation property
$\mathbf{R}(\mathbf{C})$	:	Range of C where C is a canonical embedding
$\mathbf{X}^{\perp}$	:	Orthogonal space of X or Annihilator Space
$\mathbf{\Delta_u}$	:	The algebraic notion of u-ideals
BAP	:	Bounded Approximation property
$\mathbf{B}(\mathbf{E})$	:	Banach algebra of all operators on Banach Space E
$\mathbb{A}$	:	Banach Algebra
$\mathbf{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbf{X})$	:	Banach module of continuous homomorphisms
$\mathbf{H}(\mathbf{X}, \mathbf{Y})$	:	Set of Hermitian operators from Banach space X to Y
TVS	:	Topological vector space
$(\mathbf{F}, \tau)$	:	Frechet Space F endowed with vector valued topology τ
$\mathbf{BL}(\mathbf{E}, \mathbf{X})$	:	Vector space of all bounded linear mappings from E to X
$\mathbf{Ban_R}$	:	Category of Banach modules that have i_{th} bounded morphisms
$\overline{\mathbf{Ban_R}}$	:	Category of non archimedean Banach modules with bounded morphisms
BTA	:	Composite operator
$\mathbf{Ba}(\mathbf{X})$	:	Baire-one function
$\mathbf{HB}(\mathbf{X}, \mathbf{Y})$	:	Hahn Banach extension operators from X to Y
lcs	:	Locally Convex Spaces
UPI	:	Unconditional Partition Identity

CHAPTER ONE

INTRODUCTION

1.1 Introduction

In this chapter, we provide the background information to the study, the preliminaries and basic concepts used in the study, the statement of the problem, research objectives and methods of study.

1.2 Background information

Ideals and Modules play an important role in the structure theory of rings and algebras. For instance, as an implication of the celebrated Wedderburn-Artin theorem, which is originally due to Cartan, a finite dimensional unital algebra over $\mathbb{C}$, is semi-simple if and only if it is a matrix algebra $\oplus_{i=1}^{m} M_{n_i}$ and viewed as a module with hereditarily. These structures occur naturally in algebras, for example, the kernel of a homomorphism is a two-sided ideal. In functional analysis, closed ideals are an important tool for the study of C^*-algebras. The operator space structures and the the Algebra of Ideals/Modules has progressively impacted on various findings. Some more elaborate details regarding this subject can be found in among other references [9, 16, 48, 55, 70] and most recently [67]. Mathews[16] investigated algebraic questions about the structure of $B(E)$ and ideals thereof, where $B(E)$ is the Banach algebra of all operators on a Banach space E. The study showed that there exist many examples of reflexive Banach spaces E such that $B(E)$ is not Arens regular. In the Banach Algebra setting, the study determined classes of modules in a C^*- algebra that admit the Arens's product. The approximate properties, the nuclear operators and integral properties of such modules were determined. Linus[48] studied the ideals and boundaries in the Algebras of holomorphic functions. In particular, the study investigated the spectrum of certain Banach algebras. Properties like generators of maximal ideals and generalized Shilov boundaries were studied. In particular it was shown that if the $\delta-$equation has solutions in the algebra of bounded functions or continuous functions up to the boundary of a domain $D \subset\subset \mathbb{C}^n$, then every maximal ideal over D is generated by the coordinate functions. This implies that the fibres over D in the spectrum are trivial and that the projection on $\mathbb{C}^n$ of the $n-1$ order generalized Shilov boundary is contained in the boundary of D. The complex analytic theory propagated was used in the determination of the generators of functional algebras in manifolds.

For the unifications of different $s-$numbers, an analogy of axiomatic theory of s numbers in Banach spaces is readily available in literature (cf. [74]). Previously, Rhoades[65] had generalized the classes of operators of l_p type and operators of Cesaro type by introducing an arbitrary infinite matrix $A = (a_{nk})$ using approximation numbers of a bounded linear operator. Alfsen and Efros[6] improved the notion of two-sided ideals to Banach spaces, where they introduced M-ideals. Sonia[70] enriched the theory of one sided $M-$structure of operator spaces and operator algebras in the setting meant to develop a non-commutative perspective of operator spaces which are one-sided $M-$ ideals in their bidual. The work also investigated the $M-$ideal structure of the Haagerup tensor product of operator algebras. Further, it considered operator algebras which are, in some sense, a generalization of the algebra of the compact operators. These are called the $1-$matricial algebras. Using the Haagerup tensor product and the $1-$matricial algebra, therefore, the work constructed a variety of examples of operator spaces and operator algebras which are one-sided $M-$ideals in their bidual. The findings here however did not generalize these notions in all the operator spaces, particularly with respect to $r-$ideals and $l-$ideals. There was a good deal of development of the Theory of Multipliers which is very applicable to the Theory of modules in Banach Algebraic setting.

Let $\mathbb{A}$ be a Banach Algebra. In [59, 60, 61] Rieffel made elaborate studies of the Banach module $Hom_{\mathbb{A}}(\mathbb{A}, X)$ of continuous homomorphisms. Further results in this direction have been obtained by Sentilles and Taylor[68] and Ruess[66] in their study of the general strict topology. Related studies are attributed to among others, Shantha[71] whose work focussed on homomorphisms in the case of locally convex modules. It would be imperative to investigate the extent to which some of the findings mentioned hold in the non-locally convex setting of topological modules in Frechet spaces.

An algebra $\mathbb{A}$ over $\mathbb{K}$ with a topology τ is a topological algebra if it is a topological vector space (TVS), that is, the Frechet space in which multiplication is separately continuous[61, 77]. These spaces are considered to be complete metrizable topological algebra, in this case the multiplication is jointly continuous by Arens' Theorem [[49], p. 24]. A net $\{e_\alpha : \alpha \in I\}$ in a topological algebra $\mathbb{A}$ is called a left approximate identity (respectively right approximate identity, two-sided approximate identity) if, for all $a \in \mathbb{A}$, $lim_\alpha e_\alpha a = a$ (respectively $lim_\alpha a e_\alpha = a$, that is $lim_\alpha e_\alpha a = lim_\alpha a e_\alpha$, $\{e_\alpha : \alpha \in I\}$ is said to be uniformly bounded if there exists $r > 0$ such that $\{(\frac{e_\alpha}{r})^n : \alpha \in I;\ n = 1, 2, \cdots\}$ is a bounded set in $\mathbb{A}$ [39]. A TVS (E, τ) is called ultrabarrelled if any linear topology τ' on E, having a base of neighbourhoods of 0 formed of $\tau-$closed sets, is weaker than τ. The Frechet space (E, τ) is called ultrabornological if every linear map from E into any TVS which takes bounded sets into bounded sets is continuous[31]. Thus, every Baire TVS, in particular, $F-$space is ultrabarrelled and every metrizable TVS is ultrabornological. There is a rich theory of operator modules in $F-$spaces with limited focus to the induced ideal properties.

The $M-$embedments, ones-sided structures, multipliers and related theories of r and l ideals were developed by Sonia[70]. The main idea here was to enrich the non-commutative attributes and a generalization of ideal structures to specified operator spaces. Recently, Bence[9] developed the clarity of Algebras of operators on Banach spaces, and homomorphisms thereof. The study was devoted to the homomorphisms and perturbations of homomorphisms of such algebras with a keen focus on perturbations of homomorphisms between Banach algebras. Indeed, the finiteness and stable rank of algebras of operators on Banach spaces were determined. The results of the study showed that it is possible to develop a unified theory of maps and functors over modules. Using the methods proposed by Bence[9], Saeid[67] characterized the properties of $\lambda-$continuous functions in vector valued topological spaces. This justifies the consistent development of the the Theory of Maps with respect to the hereditary algebras in Frechet spaces. In fact, Rahul's study in [51] on some classes of operator spaces considered two classes of operators on Banach spaces. One is the class of local isometries and the second is the class of projections which are related to isometries. The isometries guarantee the preservations of local angles and distances while projections guarantee the existence of operator ideals and modules in the general setting.

It is therefore evident that the theory of Algebra of Operator spaces has undergone tremendous development with the aim of attaining a unification. Our work seeks to classify all the operator ideals, their extensions, variants and associated modules up to isomorphism in Operator Spaces. The kernels, tensors and functors of modules in extensional spaces demonstrate the rich interplay between Algebra and Analysis.

1.3 Preliminaries and Basic Concepts

In this section, some standard concepts and definitions commonly used in the sequel are given .

Definition 1.3.1. ***A left ideal in an algebra*** *A is a vector subspace I of A such that for all $a \in A, b \in I \implies ab \in I$.*

Definition 1.3.2. ***A right ideal in an algebra*** *A is a vector subspace I of A such that for all $a \in A, b \in I \implies ba \in I$.*

Definition 1.3.3. ***An ideal in an algebra*** *A is a vector subspace that is simultaneously a left and a right ideal of A.*

Definition 1.3.4. *A* ***Banach space*** *is a complete normed space.*

Definition 1.3.5. *A closed unit ball of* X *denoted* B_X *is* $\{x \in X : \|x\| \leq 1\}$.

Definition 1.3.6. *For a Banach space* X *, the dual space of* X *(or, simply, the dual of* X*) denoted by* X^**, is the Banach space* $L(X, \mathbb{F})$*, where* $\mathbb{F}$ *is the underlying scalar Field. Here* x^* *will denote an arbitrary member of* X^*.

Definition 1.3.7. *If* X *and* Y *are Banach spaces, then a linear map* $T : X \longrightarrow Y$ *is said to be abounded linear operator (or, simply an operator) if there is a positive number* M *such that* $\|Tx\| \leq M\|x\|$ *for each* $x \in X$*. The operator norm of* T *being the quantity* $\|T\| = \sup_{x \in B_X} \|Tx\|$.

Definition 1.3.8. *A bounded linear operator* $T : X \longrightarrow Y$ *between normed vector spaces* X *and* Y *is said to be a* ***contractive projection*** *if its operator norm* $\|T\| \leq 1$.

Definition 1.3.9. *Let* X *and* Y *be Banach Spaces. Then the subspace* $K(X, Y)$ *is an ideal in* $L(X, Y)$ *if* $K(X, Y)$ *is a kernel of contractive projection* P *in* $(X, Y)^*$*. That is* $P : X^* \longrightarrow Y^*$ *such that* $X^{\perp}y^* \in Y^* : y^*(x) = 0, x \in X$*. Moreover,* $K(X, Y)$ *is a u-ideal in* $(L(X, Y), \|.\|)$ *if* $\|I - 2P\| \leq 1$.

Definition 1.3.10. ***Isometry:*** *A linear transformation* T *in Banach space* X *is said to be an isometry if* $\|Tx\| = \|x\|$, $x \in X$.

Definition 1.3.11. ***Reflexive Space:*** *A Banach space* X *is said to be reflexive if* $R(C) = X^{**}$ *where* $C : X \longrightarrow X^{**}$ *is a canonical embedding of* X *into* X^{**}.

Definition 1.3.12. ***Isomorphism:*** *An isomorphism of a Banach space* X *onto a Banach space* Y *is a bijective linear mapping that preserves the norm, that is* $\|Tx\| = \|x\|, x \in X, T : X \longrightarrow Y$.

Definition 1.3.13. ***Separable:*** *A topological space* (X, τ) *is said to be separable if there exists a countable subset* A *of* X *such that* $\bar{A} = X$*, that is* A *is dense in* (X, τ)*, where* τ *is a collection of all open subsets of* X.

Definition 1.3.14. *L-**Summand:** Closed subspace X of a Banach space Y is said to an L-Summand (respectively M-Summand) if there exists a closed subspace $X^{\perp}$ of Y such that $Y = X \oplus X^{\perp}$ and satisfies the norm condition $\|x + x'\| = \|x\| + \|x'\|$ respectively, $\|x + x'\| = max(\|x, \|x'\|)$ for all $x \in X$ and $x' \in X^{\perp}$.*

Definition 1.3.15. *L_p-**Summand:** A closed subspace X of a Banach space Y is said to an L_p-Summand $(1 \leq p \leq \infty)$ if there exists a projection P in X such that $P(X) = Y$ and for all $x \in X$ we have $\|x\|^p = \|Px\|^p + \|x - Px\|^p$.*

Definition 1.3.16. ***M-ideals:** A closed subspace X of a Banach space Y is an M-ideal if its annihilator $X^{\perp}$ in Y^* is $L_1 - Summand$ that is $\|x\| = \|Px\| + \|x - Px\|, p = 1$.*

Definition 1.3.17. ***u-ideals** are M-ideals defined on a real Banach space. If X is a u-ideal then the induced projection $P : Y \longrightarrow X$ with $P(Y) = X$ and $kerP = Z$ where Z is u-complement of X satisfies $\|I - 2P\| = 1$.*

Definition 1.3.18. *A closed subspace X of a Banach space Y is called a strict u-**ideal** in Y if there exists a linear projection P on Y^* with $kerP = X^{\perp}$ such that $\|I(Y^*) - 2P\| = 1$ and the range P of an induced projection on Y^{***} is a norming subspace of Y^*.*

Definition 1.3.19. *The projection $P : Y^* \longrightarrow Y^*$ is said to be **Hermitian** if and only if $\|I - (1 + \alpha)P\| = 1$ whenever $|\alpha| = 1,$ where $\alpha \in \mathbb{C}$.*

Definition 1.3.20. ***Dual Space:** is a space of all continuous linear functions on real or complex Banach space. That is $X^* = L(X, \mathbb{F})$ where $\mathbb{F})$ is a scalar field.*

Definition 1.3.21. *We say that a Banach space X is a h-**ideal** in Y if there exists a Hermitian projection $P : Y^* \longrightarrow Y^*$ with $kerP = X^{\perp}$.*

Definition 1.3.22. *The weak topology on X, denoted by $\delta(X, X^*)$ is the smallest topology on X for which x^* is continuous.*

For the weak topology, the net (x_d) in X converges to some x in X whenever $lim(x_d, x^*) = (x, x^*)$ for each $x^* \in X^*$. In this case we say (x_d) converges weakly to x.

Definition 1.3.23. *Given x in X, consider the functional f_x on X^* given by $f_x(x^*) = (x, x^*)$, then the weak* topology on X^*, denoted by $\delta(X^*, X)$ is the smallest topology on X^* for which each of the functionals f_x is continuous. For the weak* topology then a net $(x_d)^*$ in X^* converges to some x^* in X^* whenever $\lim_d (x, x_d)^* = (x, x^*)$ for each x in X.*

Definition 1.3.24. *Let F be a linear subspace of a Banach space E. A linear operator $\Phi : F^* \longrightarrow E^*$ is called a **Hahn-Banach extension operator** if $(\delta f^*)(f) = f^*(f)$ and $\|\delta f^*\| = \|f\|$ for all $f \in F$ and $f^* \in F^*$.*

Definition 1.3.25. *If $i : F \longrightarrow E$ is an inclusion mapping and $\Phi \in B(F, E)$ then the projection P defined on E^* by $P(e^*) = \Phi(i^*(e^*))$, $e^* \in E^*$ and $kerP = F^{\perp}$ has norm one and $kerP = F^{\perp}$ where P is called an **ideal projection**.*

Definition 1.3.26. *A subspace X of a Banach space Y is said to have property U in Y if every $x^* \in X^*$ has a unique norm-preserving extension $y^* \in Y^*$.*

Definition 1.3.27. *A Banach space X is said to have the **compact approximation property** if there exists a net (T_α) in $K(X)$ such that $T_\alpha x \longrightarrow x$ for all $x \in X$. If the net (T_α) in $K(X)$ can be chosen to be $\|T_\alpha\| \leq 1$ for all α, then we say that X has the metric compact approximation property.*

Definition 1.3.28. *A Banach space X is said to have the **unconditional metric approximation property** (respectively unconditional metric compact approximation property) if there exists a net $(T_\alpha) \subset (X, X)$ (respectively $K(X, X)$) with $\limsup_\alpha \|I - 2T_\alpha\| \leq 1$ such that $T_\alpha x \longrightarrow x$ for all $x \in X$.*

Definition 1.3.29. *A Banach space X is said to have the **Radon-Nikodym property** with respect to μ if every bounded linear operator $T : L^1(\mu) \longrightarrow X$ is such that there exists $g \in L^1(\mu)$ with $\|g\| = \|T\|$.*

Definition 1.3.30. *Let X and Y be Banach spaces. A linear transformation $T \mu L(X, Y)$ is compact if for any bounded sequence (x_n) in X, the sequence (Tx_n) in Y contains a convergent subsequence.*

Definition 1.3.31. *An operator T in $L(X, Y)$ is said to be of a finite rank if it has a finite dimensional range. It turns out that an operator $T : X \to Y$ of finite rank can be written in the form $T = \sum_n \langle x, x_n{}^* \rangle y_n$ for some finite sequence $x_1{}^*, \ldots\ldots\ldots x_m{}^*$ in X^* and $y_1, \ldots\ldots\ldots y_m$ in Y.*

Definition 1.3.32. *Consider the collection L of operators between Banach spaces that is the class $L = \{T : T \in L(X, Y)\}$ for some Banach spaces X, Y. For some sub collection J of L, let $J(X, Y)$ denote the collection $J \cap L(X, Y)$. A sub collection J of L is said to be an operator ideal if J possesses the following properties:*

(i) If X and Y are Banach spaces, then $J(X, Y)$ is a subspace of $L(X, Y)$ which contains $F(X, Y)$.

(ii) The ideal property: If W, X, Y, Z are Banach spaces and R is in $L(Y, Z)$, if T is in $L(X, Y)$ and S is in $L(W, X)$, then RTS is in $J(W, Z)$.

1.4 Statement of the Problem

The structure Theory of Operator Ideals and Modules in Banach spaces has demonstrated their fundamental importance and applications in Algebraic Geometry, Quantum Mechanics, Category Theory and Algebra, hence opening up more research aimed at their unifications. These structures are significant in the study of structures of rings and algebras. For instance, as an implication of the celebrated Wedderburn-Artin Theorem, which is originally due to Cartan, a finite dimensional unital algebra over $\mathbb{C}$, is semi-simple if and only if it is a matrix algebra $\oplus_{i=1}^{m} M_{n_i}$ and viewed as a module with hereditarity. Ideals and Modules occur naturally in algebras, for example, the kernel of a homomorphism is a two-sided ideal. In functional analysis, closed ideals are an important tool for the study of C^*-algebras. Classes of ideals in Banach Spaces characterized using certain projections have been widely determined in literature by various researchers. The operator ideals in the L and M class and their variants, having restrictions are available. The $M-$embedments, ones sided structures, multipliers and related theories of r, $l-$ideals were developed by Sonia[70] with a hope to enrich the non-commutative attributes and a generalization of ideal structures to specified operator spaces while Bence[9] developed the clarity of Algebras of operators on Banach spaces, and homomorphisms thereof. Despite the fact that studies concerning the Algebra of Ideals and Modules in operator spaces as well as their applications is still active, their general classification and extension remain unsettled. We therefore sought to characterize and classify Ideals and Modules in various operator spaces including: the Banach space, Hahn- Banach space, Frechet Spaces and Topological Algebras with an view to generalizing their intrinsic characteristics and relationships in the mentioned spaces.

1.5 Research Objectives

1.5.1 Main Objective

To characterize the Algebra of Ideals and Modules in Operator Spaces.

1.5.2 Specific Objectives

The specific objectives of this research study were;

(i) To determine the classification of ideals in the set of operators in Banach spaces.

(ii) To characterize some spaces of ideal operators and ideal extensions to Frechet spaces.

(iii) To extend the approximation properties of the ideals through the Integral and Nuclear Operators.

(iv) To determine the algebra of Banach Modules and Functors over the Frechet Spaces.

1.6 Significance of the study

The applications of the theory of Banach ideals has been numerous, but mainly in three directions: Measure Theory on Banach spaces, Structure Theory of Banach spaces and classifying types of locally convex spaces for instance Schwartz spaces, nuclear spaces. The theory of ideal of operators has its beautiful applications in nuclear and integral spaces. This study opens an interesting phenomenon that the intrinsic characteristics of classes of operators considered rather than the structure of the underlying spaces is sufficient to determine the ideal properties. The topological interplay between ideals and modules which allows the hereditary properties of ideals to be used to study modules opens an area of interest in algebra. Furthermore, the results of this study display the interplay between Algebra and Analysis, hence contributing to the body of knowledge in both disciplines.

1.7 Methods of Study

The following methods have been used in the study:

i. **The Radon-Nikodym property**

   Theorem 1.7.1. *Let (Ω, Σ) be a σ-algebra, let X, Y, Z be Banach spaces, and let $\nu : \Sigma \to B(X, Y)$ be an OVM. Then*
   there is an isometric isomorphism between $\left(X \hat{\otimes}_\pi Y^*\right)^*$ and $B\left(X, Y^{**}\right)$ so we have an identification $B\left(X, Y^{**}\right) = \left(X \hat{\otimes}_\pi Y^*\right)^*$ through which the action of an operator $T \in B\left(X, Y^{**}\right)$ as a linear functional on $X \hat{\otimes}_\pi Y^*$ is given by $\langle x \otimes \psi, T\rangle = \langle \psi, Tx\rangle$;

ii. **Bounded Approximation Properties**

a. Godfrey, Kalton, and Saphar [24]: Let X be a separable reflexive Banach space and Y any Banach space. If X has (UKAP), then $K(X, Y)$ is a $u-$ideal in $L(X, Y)$.

b. Sonia [70]: Suppose X^* has the completely bounded approximation property and X is a locally reflexive operator space. Then X has the completely bounded approximation property.

iii. **Continuity and hypocontinuity criteria for modules and functors**

   Rieffel[61]:

a. Let $\mathbb{A}$ be a Banach algebra and V an $\mathbb{A}-$module. Suppose that there is a constant, M, such that for every $a \in A$, every finite collection $V_i, \ldots v_k$ of elements of V, and every $\varepsilon > 0$, there exists an element $e \in A$ such that $\|e\| \leq M$ $\|a - ae\| < \varepsilon$ and $\|a - ae\| < \varepsilon$ for $1 \leq j \leq k$. Then for every sequence v_n of elements of V which converges to 0 there exists $e \in A$ and a sequence w_n of elements of V which converges to 0 such that $v_n = aw_n$ for all n.

b. Let $\mathbb{A}$ be a Banach algebra with bounded approximate identity and continuous involution. Then every positive linear functional on $\mathbb{A}$ is continuous.

CHAPTER TWO

LITERATURE REVIEW

In this chapter, we provide detailed survey of literature concerning Banach space ideals and operator ideals, exploring their structures, Banach Rings and Modules, Frechet spaces, Modules and Maps.

2.1 Banach Space Ideals

There are enormous applications of Ideals and Modules in Spectral Theory, Geometry of Banach spaces, Theory of eigenvalue distributions among others, that have necessitated their studies to occupy special importance in functional analysis. Many useful operator ideals have been defined by using sequence of $s-$numbers. For the unifications of different $s-$numbers, an analogy of axiomatic theory of $s-$numbers in Banach spaces is readily available in Literature (cf. [74]). Previously, Rhoades[65] generalized the classes of operators of l_p type and operators of Cesaro type by introducing an arbitrary infinite matrix $A = (a_{nk})$ using approximation numbers of a bounded linear operator. Alfsen and Efros[6] generalized the notion of two-sided ideals to Banach spaces, where they introduced M-ideals. The main idea was to generalize the two-sided ideals in a C^*algebra and obtained a variant which would serve as a tool for the study of Banach spaces. The notion of M-ideals is an appropriate generalization, since in a C^*-algebra, $M-$ideals coincide with the two-sided closed ideals[69].
Although they generalized the notion of ideals to Banach spaces, they did not determine the classification of ideals in the set of operators in Banach spaces.

Indeed, the subject of operator ideals and their characterization has been a subject of interest for quite some time now. Some of the advancements in this direction are attributed to Godefroy, Kalton, and Saphar [24]. In their work on unconditional ideals in Banach spaces they extended the notion of an ideal by relaxing some conditions and stated that if X is a subspace of a Banach space Y then X is said to be an ideal in Y if $X^{\perp}$ is the kernel of a contractive projection on Y^*. The study in [24] gives a general analogy of h-ideals and u-ideals. It is shown that if a separable Banach space X is an h-ideal in X^{**} then X has a complex form of Pelczynski's property (u) with constant one and the Baire-one function $Ba(X)$ in X^{**} are complemented by an Hermitian projection and the converse holding under a compatibility condition which is a necessity. This idea was related to the more familiar M-ideal and to the Banach lattices. Motivated by some ideas of Godun[25] they introduced the Godun set of a Banach space $X,$ dented by $G(X)$ and defined as a set of all scalars λ such that $\|I - \lambda\pi\| = 1$ where π is the canonical projection of X^{***} onto X^*. If X contains a copy of l_1, then the Godun set $G(X)$ reduces to $\{0\}$. If X is separable and X^* is non separable then $G(X) \subset [0,1]$. When X^* is separable and $1 < \lambda < 2$ it is shown that X can be renormed so that $[0,\lambda] \subset G(X)$. In this direction it is shown that a Banach space with separable dual can be renormed to satisfy hereditarily an "almost" optimal uniform smoothness condition. This optimal condition occurring when the canonical decomposition $X^{***} = X^{\perp} \oplus X^*$ is unconditional. From the studies in [24] we have the following:

Proposition 2.1.1. *Let X be a separable Banach space so that $G(X)$ contains some $\lambda > 1$ or $\|I - 2\pi\| < 2$. Then X^* is separable.*

The result above can also be restated as follows in the converse.

Theorem 2.1.1. *Let X be a separable Banach space for which X^* is separable. If $1 < \lambda < 2$ then X can be equivalently normed so that $\lambda \in G(X)$.*

For a Banach Space X and a closed subspace M of X^*, the characteristic $r(M)$ of M is the greatest constant r such that $\sup |x^*(x)| \geq r\|x\|, x^* \in M, \|x^*\| \leq 1$. Therefore, if X is a separable Banach space with separable dual and if $\varepsilon > 0$, then X can be equivalently renormed so that any subspace Z of a quotient space of X has the property that when M

is a proper closed subspace of Z^* then $r(M) \leq \frac{1}{2}+\varepsilon$. The results improved the main result of Finet, Schachermayer[20]. But a question still arose: if X^* is separable does there exist an equivalent norm on X so that any proper closed subspace of X^* has characteristic at most $\frac{1}{2}$? The answer is positive for strict u-ideals, however it need not be true always for example in Godun[25], it is shown that such a norm exists when X is a quasi-reflexive of order one. Motivated by the findings in Godun[25], Finet, Schachermayer[20] determined a subspace X of a Banach space Y to be an u-ideal if there is a projection P with $\|I-2P\| = 1$ on Y^* with kernel $X^{\perp}$ and it is h-ideal if there exists a Hermitian projection P with $\|I-2P\| = 1$ on Y^* with kernel $P = X^{\perp}$.

Further investigations where for a separable Banach space, the idea of compact operators $K(X)$ is a u-ideal or an h-ideal in $L(X)$ or $K(X)^{**}$ was done. For example it is shown that $K(X)$ is an h-ideal in $K(X)^{**}$ if and only if X has "unconditional compact approximation property" and X is an M-ideal. This work opened doors to a lot of interest in the area for further research. Rao[62] studied the ideals in Banach spaces and showed that for a Banach space X and an ideal Y in X, the injective tensor product space $Y \otimes_{\varepsilon} Z$ is an ideal in $X \otimes_{\varepsilon} Z$ for any Banach space Z. This as a consequence gave a way of proving some known results about intersection properties of balls and extensions of operators on injective tensor product spaces in a unified way. An example of a Banach space X such that $K(X,X)$ is not an ideal in $K(X, X^{**})$ was given in works of Lima, and Oja [44] on ideals of compact operators. In this work it is shown that if z^* is a weak * denting point in the unit ball of Z^* and if X is a closed subspace of a Banach space Y, then the set of norm-preserving extensions $HB(x^* \otimes z^*) \subseteq L(Z^*, Y)^*$ of a functional $(x^* \otimes z^*) \in (Z \otimes X)^*$ equals the set $HB(x^*) \otimes \{z^*\}$. Using this result, it is shown that if X is an M-ideal in Y and Z is a reflexive Banach space, then $K(Z,X)$ is an M-ideal in $K(Z,Y)$ whenever $K(Z,X)$ is an ideal in $K(Z,Y)$ and this led to the result that $K(Z,X)$ is an ideal (resp. an M-ideal) in $K(Z,Y)$ for all Banach spaces Z whenever X is an ideal (resp. an M ideal) in Y and X^* has the compact approximation property with conjugate operators [52].

Rao's work in [63] on intersection of ideals in Banach space picked interest in studying finite intersections of ideals in Banach spaces. It is shown that for a Banach space X, if in the bidual X^{**}, every ideal of finite codimension is the intersection of ideals of codimension one, then the same property holds in X. Further showed that if a Banach space whose dual is isometric to $L^1(\mu)$ for a positive measure μ then any ideal of finite codimension is a finite intersection of ideals of codimension one. Moreover, Rao[64] introduced the notion of an extremely strict ideal. In particular, the study showed that the space of affine continuous functions on $\mathbb{K}$ is an extremely strict ideal in the space of continuous functions on $\mathbb{K}$. For injective tensor product spaces, a cancelation Theorem for extremely strict ideals was proved and non-reflexive Banach spaces which are not strict ideals in their fourth dual exhibited.
However, they did not characterize this spaces of ideal operators and the extensions to the Frechet spaces.

The study of Abrahamsen et al [3] on unconditional ideals of finite rank operators gave characterizations of when $F(Y, X)$ is a u-ideal in $W(Y, X)$ for every Banach spaces X and Y in terms of nets of finite rank operators approximating weakly compact operators. Similar characterizations were given for the cases when $F(Y, X)$ is a u-ideal in $W(Y, X)$ for every Banach space Y, when $F(Y, X)$ is a u-ideal in $W(Y, X)$ for every Banach space Y and when $F(Y, X)$ is a u-ideal in $K(Y, X)$ for every Banach space Y. Abrahamnsen et al[3] defined and studied λ-strict ideals in Banach spaces in which for $\lambda = 1$ means strict ideals. Strict u-ideals in their biduals are known to have the unique ideal property and the study in [3] revealed that the λ-strict u-ideals also have unique properties in their biduals, at least for $\lambda > 1/2$.

Lima et al [47] on the Geometry of operator spaces considered bounded approximation properties via nuclear and integral operators. Starting with a Banach space X and a Banach operator A, they deteremined the λ bounded approximation property for A ($\lambda - BAP$ for A) and showed that for every Banach space Y and every Operator $T \in A(X, Y)$, there exists a net $(S\alpha)$ of finite rank operators on X such that $S\alpha \to I_X$ uniformly on compact subsets of X and $\limsup \|TS\alpha\|_{\mathrm{A}} \leq \lambda \|T\|_{\mathrm{A}}$. They further proved that the weak λ-BAP is precisely the λ-BAP for the ideal N of nuclear operators. Lima [43] conducted a study on the metric approximation properties in Banach spaces where it was shown that if a Banach space Y is a u-ideal in its bidual Y^{**} with respect to the canonical projection on the third dual Y^{***}, then Y^* contains "many" functionals admitting a unique norm-preserving extension to Y^{**} and the dual unit ball B_{Y^*} is the norm-closed convex hull of its weak * strongly exposed points. Consequently, Martsinkevits and Poldvere[50] in their study on the structure of the dual unit ball of strict $u-$ ideals showed that if Y is a strict u-ideal in a Banach space X with respect to an ideal projection P on X^*, and X/Y is separable, then $B_{Y^*}(X)$ is the τP closed convex hull of functionals admitting a unique norm-preserving extension to X, where τP is a certain weak topology on Y^* defined by the ideal projection P. A question that arises still is: if X is a Banach space which is a strict u-ideal in its bidual and Y any separable subspace of X, then is Y a strict u-ideal in its bidual?, and is X separably determined?. Our study provides a partial solution to this question.

Lima et al[41] developed a Compact Approximation Theory where they showed that a Banach space X has the compact approximation property if and only if for every Banach space Y and every weakly compact operator T : Y $\to X$, the space $\mathfrak{S} = \{S \circ T : S$ is a compact operator on X $\}$ is an ideal in $\mathfrak{J} = \mathrm{span}(\mathfrak{S}, \{T\})$ if and only if for every Banach space Y and every weakly compact operator T : Y $\to X$, there is a net (S_γ) of compact operators on X such that $\sup_\gamma \|S_\gamma T\| \leq \|T\|$ in the strong operator topology. Similar results for dual spaces were also shown. Now, let $X \subseteq Y$ be Banach spaces and let A $\subseteq$ B be closed operator ideals. Let Z be a Banach space having the Radon-Nikodym property. Lima, and

Oja [44] showed that if $\Phi : \mathrm{A}(Z,X)^* \rightarrow \mathrm{B}(Z,Y)^*$ is a Hahn-Banach extension operator, then there exists a set of Hahn-Banach extension operators $\phi_i : X^* \rightarrow Y^* i\ i \in I$, such that $Z = \sum_{i \in I} \oplus_i Z_{\Phi\phi_i}$, where $Z_{\Phi\phi_i} = \{z \in Z : \Phi\left(x^* \otimes z\right) = \left(\phi_i x^*\right) \otimes z, x^* \in X^*\}$ [54, 72]. Further if $\mathrm{B}(\mathrm{Z}, \hat{Y})$ is an ideal in $\mathrm{B}(\widehat{Z}, Y)$ for all equivalently renormed versions $\widehat{Z}$ of Z, then there exist Hahn-Banach extension operators $\Phi : \mathrm{A}(Z,X)^* \rightarrow \mathrm{B}(Z,Y)^*$ and $\Phi : X^* \rightarrow Y^*$ such that $Z = Z_{\Phi\phi}$.

Hamard and Lima [26] investigated Banach spaces X such that X is an M-ideal in X^{**}. Subspaces, quotients and c_0-sums of spaces which are M-ideals in their biduals are again of this type. A non-reflexive space X which is an M-ideal in X^{**} contains a copy of c_0. In their study, they showed that if $K(X)$ is an M-ideal in $L(X)$, then X is an M-ideal in X^{**}. Also, if X is reflexive and $K(X)$ is an M-ideal in $L(X)$, then $K(X)^{**}$ is isometric to $L(X)$, that is, $K(X)$ is an M-ideal in its bidual. Moreover, for real such spaces, $K(X)$ contains a proper M-ideal if and only if X or X^* contains a proper M-ideal. The proofs of these results are based upon the fact that X is an M-ideal in X^{**} if and only if the natural projection from X^{***} onto X^* is an L-projection. Using local reflexivity it is shown that if X is an M-ideal in X^{**} and X is non-reflexive, then X contains almost isometric copies of c_0. From this it follows that subspaces and quotients are isomorphic to dual spaces are reflexive.

Lima [42] studied strict u-ideals in Banach spaces. A Banach space X is a strict u-ideal in its bidual when the canonical decomposition $X^{***} = X^* \oplus X^{\perp}$ is unconditional. In characterizing Banach spaces which are strict u-ideals in their bidual it is shown that if X is a strict u-ideal in a Banach space Y then X contains c_0. It is also shown that ℓ_∞ is not a u-ideal. Let X be a subspace of a Banach space Y, X is said to be a summand of Y if it is the range of a contractive projection and that X is an ideal in Y if $X^{\perp}$ is the kernel of a contractive projection on Y^*.

A norm one operator $\phi : X^* \to Y^*$ such that $\phi(x^*)(x) = x^*(x)$ is said to be a Hahn-Banach extension operator. The set of all such ϕ is denoted by $\mathrm{B}(X,Y)$. For every $\phi \in \mathrm{B}(X,Y)$ we have

$$Y^* = X^\perp \oplus \phi(X^*).$$

Let i_X be the natural embedding $i_X : X \to Y$. $P_\phi = \phi \circ i_X^*$ is a norm one projection on Y^* with $\ker P = X^\perp$. X is an ideal in Y if and only if $\mathrm{B}(X,Y) \neq \emptyset$. If $\left\|x^\perp + \phi(x^*)\right\| = \left\|x^\perp - \phi(x^*)\right\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$, then X is a u-ideal in Y and that ϕ is unconditional if and only if $\|I - 2P_\phi\| = 1$ which gives a well-known notion of an M-ideal [23, 27] if $\left\|x^\perp + \phi(x^*)\right\| = \left\|x^\perp\right\| + \|\phi(x^*)\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$.
However, they did not extend these approximation properties of the ideals through the Integral and Nuclear Operators.

The operator space structures and the the Algebra of Ideals/Modules has progressively impacted on various findings. Some more elaborate details regarding this subject can be found in among other references ([16][48][70][55] [9]) and most recently [67]. Mathews[16] investigated algebraic questions about the structure of $B(E)$ and ideals thereof, where $B(E)$ is the Banach algebra of all operators on a Banach space E. The study showed that there exist many examples of reflexive Banach spaces E such that $B(E)$ is not Arens regular. In the Banach Algebra setting, the study determined classes of modules in a C^*- algebra that admit the Arens's product. The approximate properties, the nuclear operators and integral properties of such modules were determined. Linus[48] studied the Ideals and Boundaries in Algebras of Holomorphic Functions. In particular, the study investigated the spectrum of certain Banach algebras. Properties like generators of maximal ideals and generalized Shilov boundaries are studied. In particular it was shown that if the $\delta-$equation has solutions in the algebra of bounded functions or continuous functions up to the boundary of a domain $D \subset\subset \mathbb{C}^n$ then every maximal ideal over D is generated by the coordinate functions. This implies that the fibres over D in the spectrum are trivial and that the projection on $\mathbb{C}^n$ of the $n-1$ order generalized Shilov boundary is contained in the boundary of D. The complex analytic theory propagated was used in the determination of the generators of functional algebras in manifolds.

The $M-$embedments, ones sided structures, multipliers and related theories of $r-$ ideals and $l-$ideals were developed by Sonia[70]. The main idea here was to enrich the non-commutative attributes and a generalization of ideal structures to specified operator spaces. Recently, Bence[9] developed the clarity of Algebras of operators on Banach spaces, and homomorphisms thereof. The study was devoted to the homomorphisms and perturbations of homomorphisms of such algebras with a keen focus on perturbations of homomorphisms between Banach algebras. Indeed, the Finiteness and stable rank of algebras of operators on Banach spaces were determined [75]. The results of the study showed that it is possible to develop a unified Theory of maps and functors over modules. Using the methods proposed by Bence[9], Saeid[67] characterized the properties of $\lambda-$continuous functions in vector valued topological spaces. This justifies the consistent development of the the Theory of Maps with respect to the hereditary algebras in Frechet spaces. In fact, Rahul's study in [51] on the study of some classes of operator spaces considers two classes of operators on Banach spaces. One is the class of local isometries and the second is the class of projections which are related to isometries. The isometries guarantee the preservations of local angles and distances while projections guarantee the existence of operator ideals and modules in the general setting.

2.2 Banach Rings, Frechet Modules and Continuous Maps

The works closely related to the Theory of Banach Rings and Banach Modules was determined by Gelfand, Raikov and Shilov[22] in their study on commutative normed rings. The main idea in this work was a display of the intrinsic relationship between the ring structure and the topology induced by the norm. They revealed that it was possible to characterize the module structures over the normed rings taken as a topological algebra. This was a follow up to the study by Kaplansky[37, 79] on the properties of topological rings. In particular, Kaplansky[37] determined the category of topological algebras that obey the ideal properties in the operator and the Banach modules in the vector valued topological spaces characterized.

Sonia[70] enriched the Theory of one sided $M-$structure of operator spaces and operator algebras in the setting meant to develop a non-commutative theory of operator spaces which are one-sided $M-$ ideals in their bidual. The work also investigated the $M-$ideal structure of the Haagerup tensor product of operator algebras. Further, it considered operator algebras which are, in some sense, a generalization of the algebra of the compact operators. These are called the $1-$matricial algebras. Using the Haagerup tensor product and the $1-$matricial algebra, the work constructed a variety of examples of operator spaces and operator algebras which are one-sided $M-$ideals in their bidual. The findings here however do not generalize these notions in all the operator spaces particularly with respect to $r-$ideals and $l-$ideals. There was a good deal of development of the Theory of Multipliers which is very applicable to the Theory of modules in Banach Algebraic setting [53, 57].

In [59, 60, 61] Rieffel made an elaborate study of the Banach module $Hom_{\mathbb{A}}(\mathbb{A}, X)$ of continuous homomorphisms. Further results in this direction have been obtained by Sentilles and Taylor[68] and Ruess[66] in their study of the general strict topology. Related studies are attributed to among others, Shantha[71] whose study focussed on homomorphisms in the case of locally convex modules. We purpose to investigate the extent to which some of the results of above authors are also true in the non-locally convex setting of topological modules in generalized spaces akin to Frechet spaces.

An algebra $\mathbb{A}$ (over $\mathbb{K}$) with a topology τ is called a topological algebra if it is a topological vector space (TVS) commonly called the Frechet Space, in which multiplication is separately continuous. A complete metrizable topological algebra is called an $F-$algebra; in this case the multiplication is jointly continuous by Arens' Theorem [[49], p. 24]. A net $\{e_\alpha : \alpha \in I\}$ in a topological algebra $\mathbb{A}$ is called a left approximate identity (respectively right approximate identity, two-sided approximate identity) if, for all $a \in \mathbb{A}$, $lim_\alpha e_\alpha a = a$ (respectively $lim_\alpha a e_\alpha = a$, that is, $lim_\alpha e_\alpha a = lim_\alpha a e_\alpha = a$, $\{e_\alpha : \alpha \in I\}$ is said to be uniformly bounded if there exists $r > 0$ such that $\{(\frac{e_\alpha}{r})^n : \alpha \in I;\ n = 1, 2, \cdots\}$ is a bounded set in $\mathbb{A}$. A TVS (E, τ) is called ultrabarrelled if any linear topology τ' on E, having a base of neighbourhoods of 0 formed of $\tau-$closed sets, is weaker than τ. The Frechet space (E, τ) is called ultrabornological [31] if every linear map from E into any TVS which takes bounded sets into bounded sets is continuous. Every Baire TVS (in particular, $F-$space) is ultrabarrelled. Every metrizable TVS is ultrabornological.

Let X be a Frechet space and $\mathbb{A}$ be a topological algebra, both over the same field $\mathbb{K}$. Then X is called a topological left $\mathbb{A}-$module if it is a left $\mathbb{A}-$module and the module multiplication $(a, x) \to a.x$ from $\mathbb{A} \times X$ into X is separately continuous. If $b(\mathbb{A})$ (respectively $b(X)$) denote the collection of all bounded sets in $\mathbb{A}$ (respectively X), then module multiplication given above is called $b(\mathbb{A})$-hypocontinuous (respectively $b(X)$ hypocontinuous)[49] if, given any neighbourhood G of 0 in X and any $D \in b(\mathbb{A})$ (respectively $B \in b(X)$), there exists a neighbourhood H of 0 in X (respectively V of 0 in $\mathbb{A}$) such that $D.H \in G$ (respectively $V.B \in G$). Clearly, joint continuity implies hypocontinuity which also implies separate continuity; however, the converse need not hold. If E and X are TVSs, BL(E,X) (respectively CL(E,X)) denotes the vector space of all bounded (respectively continuous) linear mappings from E into X. Clearly, $CL(E, X) \in BL(E, X)$ with $CL(E, X) = BL(E, X)$ if E is ultrabornological (in particular metrizable). A mapping $T : E \to X$ is called a topological isomorphism if T is linear and a homeomorphism. If X is a left $\mathbb{A}-$module, then $\mathbb{A}$ is said to be faithful in X if, for any $x \in X$, $a.x = 0$ for all $a \in \mathbb{A}$ implies that $x = 0$ (cf. [35],[68]).

Let E and X be topological left $\mathbb{A}$-modules, where E and X are TVSs and A is a E into X. If E is an $\mathbb{A}$-bimodule, then defining $(a * T)(x) = T\,(x\ .a)$, then, $\mathrm{Hom}_A(E, X)$ becomes a left $\mathbb{A}$-module. In fact, for any $b \in A, x \in E$

$$(a * T)(b \cdot x) = T((b \cdot x) \cdot a) = T(b \cdot (x.a)) = b \cdot T(x.a) = b \cdot (a * T)(x).$$

In particular, $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$ is a left $\mathbb{A}$-module. Note that if $\mathbb{A}$ is commutative, then defining $(T*)(x) = T(a \cdot x), \mathrm{Hom}_{\mathbb{A}}(E, X)$ becomes a right $\mathbb{A}$-module.

We note that $\mathrm{Hom}_{\mathbb{A}}(E, X)$ has been extensively studied in the case of E and X as the Banach modules of Banach valued function spaces $L^1(G, \mathbb{A})$ and $C_\circ(G, \mathbb{A})$, where G is a locally compact abelian group and $\mathbb{A}$ is a commutative Banach algebra. Abel [1] has studied it in the setting of topological bimodule algebras. If $E = X = \mathbb{A}$, then $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A})$ is the usual multiplier algebra of $\mathbb{A}$, and is denoted by $M(\mathbb{A})$. In fact, there is a vast literature dealing with the notions of left multiplier, right multiplier, multiplier and double multiplier (see [11, 29, 33, 35, 38, 56, 76]).
However, the determination of the algebra of Banach Modules and Functors remain unsettled.

From the aforementioned literature, we observe that studies concerning the Algebra of Ideals and Modules in operator spaces as well as their applications is still active. We therefore aimed to consider the structures of Ideals and Modules in various operator spaces including: the Banach space, Hahn- Banach space, Frechet Spaces and Topological Algebras.

CHAPTER THREE

CLASSIFICATION OF IDEALS IN BANACH SPACES

3.1 Introduction

This chapter focusses on the various classes of ideals in Banach spaces. Special attention is given to the properties involving the ideal properties, the metric approximation properties, the hereditary properties in relation to the ideal extensions for example the Hahn-Banach extension, projection and embedments in the biduals of the Banach Space.

3.2 Preliminary Results

An operator ideal is a special kind of class of continuous linear operators between Banach spaces. Let an operator T belong to an operator ideal J, then for any operators A and B which can be composed with T as BTA then $BTA \in J$. Indeed, J contains the class of finite rank Banach Space operators. Now given $L(X, Y)$. Then $J(X, Y) \subseteq L(X, Y)$ such that $J(X, Y) = \{T : X \longleftarrow Y : T \in J\}$. Thus an operator ideal is a subclass J of L containing every identity operator acting on a one-dimensional Banach space such that: $S + T \in J(X, Y)$ where $S, T \in J(X, Y)$. If $W, Z, X, Y \in \mathbb{K}, A \in L(W, X), B \in L(Y, Z)$ then $BTA \in J(W, Z)$ whenever $T \in J(X, Y)$.
These properties compare very well with the algebraic notion of ideals in Banach Algebras within whose classes lie compact operators, weakly compact operators, finitely strictly regular operators, completely continuous operators, strictly singular operators among others. The Radon-Nikodym property applies to separable reflexive Banach spaces with compact operators having metric approximation properties [27].

The characterization of Banach Algebra of compact operators as ideals can be demonstrated as follows:

Theorem 3.2.1. *Let X be a separable Banach Algebra of compact operators. Then:*

i. $K(X, X^{**})$ *is an ideal in* $L(X, X^{**})$

ii. $K(X, X)$ *is not an ideal in* $L(X, X)$

iii. $K(X, X)$ *is not an ideal in* $L(X, X^{**})$

Proof. Let $X = \left(\sum_{n=1}^{\infty} \oplus (Z^{**}, \|\cdot\|_n)\right)_2$ where Z^{**} is a separable Banach space and $\|\cdot\|_n$ is an equivalent norm on Z^{**}. The space X fails the metric compact approximation property, but its dual X^* has the metric approximation property. Since Z^{**} is separable, the space $(Z^{**}, \|\cdot\|_n)$ has the Radon-Nikodym property. Thus X has the Radon-Nikodym property as well (the fact that the Radon- Nikodym property is preserved under the direct sums $\ell_p(1 \leq p \leq \infty)$) and the fact that a Banach space with a bounded complete basis has the Radon-Nikodym property. Since X^* has the metric compact approximation property, by a well known result due to Johnson [35], $K(X^*, X^*)$ is an ideal in $L(X^*, X^*)$. Thus $K(X, X^{**})$ is an ideal in $L(X, X^{**})$ which establishes (i).
Since X has the Radon-Nikodym property but fails to have the metric compact approximation property, $K(X, X)$ is not an ideal in $L(X, X)$. Hence $K(X, X)$ is not an ideal in $L(X, X^{**})$. If $K(X, X)$ is not an ideal in $K(X, X^{**})$, then $K(X, X)$ is not an ideal in $L(X, X^{**})$ because $K(X, X^{**})$ is an ideal in $L(X, X^{**})$ which is impossible. Thus establishes (ii) and (iii). □

Remark 3.2.1. *From the above Theorem, the dual space X^* fails to have the metric compact approximation property with conjugate operators (although X^* has the metric approximation property). Lima [41] demonstrated that $K(Z, X)$ is an ideal in $K(Z, Y)$ for Banach space Z whenever X is an ideal in Y and X^* has the metric compact approximation property with conjugate operators. This addresses the question whether $K(Z, X)$ is an ideal in this $K(Z, X^{**})$ or not. It is clear since there is a norm one projection between the Banach space X and its dual, $K(Z, X)$ is an ideal in this $K(Z, X^{**})$ for all Banach spaces Z.*

Theorem 3.2.2. *Let X be a closed subspace of a Banach space Y. The following statements are equivalent*

i. X is an ideal in Y

ii. $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,Y)}$ for some Banach spaces Z.

iii. $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,Y)}$ for some Banach space $Z \neq \{0\}$.
In particular $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,X^{**})}$ for all Banach spaces X in Z.

Proof. $i \Rightarrow ii$. Let X be an ideal of Y and $F(X,Y)$ be a class of finite rank compact operators. Since $Z^* = L(Z,\mathbb{F})$ is a vertical distribution, then there exist $\varepsilon, s > 0$ such that $\overline{F(Z,X)}$ and $\overline{F(Z,Y)}$ can be canonically identified with $Z^* \oplus_\varepsilon X$ and $Z^* \oplus_s Y$ respectively. Thus $\overline{F(Z,X)}\Delta\overline{F(Z,Y)}$ as required so that $(i) \implies (ii)$
$ii \Rightarrow iii$. Since $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,Y)}$ for all Banach spaces Z then clearly $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,Y)}$ for some Banach space $Z \neq \{0\}$.
$iii \Rightarrow i$. Suppose $\overline{F(Z,X)}$ is an ideal in $\overline{F(Z,Y)}$. Let F be a finite dimensional subspace of Y. Let $z \in Z$ and $z^* \in Z^*$ be such that $\|z\| = \|z^*\| = z^*(z) = 1$. Let $G = \{z^* \otimes y : y \in F\} \subseteq F(Z,Y)$. Let $\varepsilon > 0$ and $V : G \to \overline{F(Z,X)}$ be an operator such that $\|V\| \leq 1+\varepsilon$ and $V(S) = S$ for all $S \in G \cap \overline{F(Z,X)}$. Now define a map $U : Y \to X$ by $U_y = (V(z^* \otimes y))\, z$. Thus we conclude that $(V(z^* \otimes y))\, z \in X$ is an ideal of Y as required. □

Proposition 3.2.1. *Let X be a closed subspace of a Banach space Y and assume that $K(Z,X)$ is an ideal in $K(Z,X^{**})$ for some Banach space $Z \neq \{0\}$. Then X is an ideal in Y if and only if $K(Z,X)$ is an ideal in $K(Z,Y)$.*

Proof. The necessity condition is standard, so we only need the 'only if' part.

Let $\phi : X^* \to Y^*$ and $\Phi : K(Z,X)^* \to K(Z,X^{**})^*$ be Hahn-Banach extension operators. Define $\Psi : K(Z,X)^* \to K(Z,Y)^*$ by $(\Psi f)(T) = (\Phi f)(\phi^* \mid Y \circ T), f \in K(Z,X)^*, T \in K(Z,Y)$. Then Ψ is linear and $\|\Psi\| \leq 1$ by definition 1.2.3. Since

$\phi^* x = x, x \in X$, we have $\phi^* \mid Y \circ T = T$ whenever $T \in (Z, X)$.Consequently, for any $T \in (Z, Y)$ and any $f \in K(Z, X)^*$, $(\Psi f)(T) = (\phi f)(T) = f(T)$, meaning that Ψf is an extension of f. Hence Ψ is a HahnBanach extension operator. $\square$

Remark 3.2.2. *$K(Z, X)$ is not an ideal in $K(Z, X^{**})$ for all X and Z unless X is a norm one projection in its bidual. However $K(X, X^{**})$ is an ideal in $K(X, X^{***})$ because X^{**} is the range of norm one projection in X^{***}.*

Theorem 3.2.3. *Let X be Banach space with the Radon-Nikodym property.Then the following statements are equivalent.*

a) X has the metric approximation property.

b) $F(X, X)[$ resp. $K(X, X)]$ is an ideal in the space $L(X, X)$.

c) $F(X, X)$ where $\alpha > 0$ and I is the identity operator (resp. $K(X, X)$) is an ideal in $\operatorname{Span}(F(X, X) \cup \{I_\alpha\})$ (resp. $\operatorname{Span} K(X, X) \cup \{I_\alpha\}$).

Theorem 3.2.4. *Let X be a Banach space .The following statements are equivalent:*

a) X has the metric approximation property (respectively the metric compact approximation property)

b) $F(Y, X)[$ resp. $K(Y, X)]$ is an ideal in the space $L(Y, X)$ for every Banach space Y.

c) $F(Y, X)$ [resp. $K(Y, X)$] is an ideal in the space $L(Y, X)$ for every separable Banach space Y.

d) $F(\hat{X}, X)[$ resp. $K(\hat{X}, X)]$ is an ideal in the space $L(\hat{X}, X)$ for every equivalent renorming $\hat{X}$ of X.

Proof. a $\Rightarrow$ b is proved in [27, 47, 80]

b $\Rightarrow$ c and b $\Rightarrow$ d because $F(Y, X)$ [resp. $K(Y, X)$] is an ideal in the space $L(Y, X)$ for every Banach space Y implies that $F(Y, X)[$ resp. $K(Y, X)]$ is an ideal in the space $L(Y, X)$ for every separable Banach space Y and $F(\hat{X}, X)$ [resp. $K(\hat{X}, X)]$ is an ideal in the space $L(\hat{X}, X)$ for every equivalent renorming $\hat{X}$ of X.

c $\Rightarrow$ a is proved as follows: Let $L \subseteq X$ be a separable subspace. It is well known that a separable ideal Y in X with $L \subseteq X$. Let $\varphi : Y^* \to X^*$. Let $\Psi : F(Y,X)^* \to K(Y,X)^*$ be Hahn-Banach extension operators and $i : F(Y,X) \to L(Y,X)$ be the inclusion map. Then $P = \Psi \circ i^*$ is an ideal projection. Let $I : Y \to X$ be the identity map. We find a net $(T_\alpha) \subseteq F(Y,X)$ with $\sup_\alpha \|T_\alpha\| \leq \|I\| = 1$ such that $x^*(T_\alpha y) \to (P(x^* \otimes y))(I) \forall y \in Y$ and $x^* \in X^*$, that is $T_\alpha \to I$ in the weak* topology. Let $\widehat{T_\alpha} = T_\alpha{}^{**} \circ \varphi^*|_x \in F(X,X)$ then $\left\|\widehat{T_\alpha}\right\| = \|T_\alpha\| \leq 1$ and $\widehat{T_\alpha}$ converges pointwise to the identity I_α on Y. It follows that X has a metric approximation property by definition.

d $\Rightarrow$ a. Let $Y = X$ and there exists $\Psi \in HB(F(X,X), L(X,X))$ such that $\Psi(x^* \otimes x) = x^* \otimes x$ for all $x^* \in X^*, x \in X$.Let $i : F(X,X) \to L(X,X)$ be an inclusive map and define $P = \Psi \circ i^*$. Using Theorem 5.4 in [40] with P as the ideal projection we conclude that X has the metric approximation property. $\square$

3.3 Ideals through the Hahn-Banach extension operator

We provide an alternative approach to the definition of an ideal through the Hahn-Banach extension operator and discuss some ideal properties under this context. The next result is well known and can be found in [15].

Lemma 3.3.1. *Let F be a closed subspace of a Banach space E. The following statements are equivalent.*

(a) F is an ideal in E.

(b) F is locally 1 -complemented in E, that is, for every finite dimensional subspace G of E and for all $\varepsilon > 0$, there is an operator $U : G \to F$ such that $\|U\| \leq 1 + \varepsilon$ and $Ux = x$ for all $x \in G \cap F$.

(c) There exists a Hahn-Banach Extension operator $\phi : F^* \to E^*$.

By defining Hahn-Banach Extension operators on tensor product spaces the following result is immediate:

Theorem 3.3.1. *Let X be an ideal in Y and let Z be an ideal in W. Then $X \otimes Z$ is an ideal in $Y \otimes W$.*

Proof. Let $\phi : X^* \to Y^*$ and $\psi : Z^* \to W^*$ be Hahn-Banach Extension operators. Let $Q : Z^{***} \to Z^*$ be the canonical projection. Using the identifications $(X \otimes Z)^* = I(X, Z^*)$ and $(Y \otimes W)^* = I(Y, W^*)$ the map $\phi : I(X, Z^*) \to I(Y, W^*)$ defined by $\phi(T) = \psi \circ Q \circ T^{**} \circ \phi^* \backslash_Y$ is clearly a Hahn-Banach extension operator since $\phi^* x = x, x \in X$, and $\psi^* z = z$ for all $z \in Z$.

□

In the sequel we consider classes of ideals where additional constrains are imposed on the projections. Hereditary properties of these classes of ideals in relation to the basic properties of a general Banach space operator ideal are also discussed.

3.4 M-ideals

There is extensive literature concerning special class of ideals known as M-ideal [14, 27, 26, 43] and decompositions of Banach spaces by means of projections satisfying certain norm conditions. What are considered as special notions are contained in the following definitions.

Definition 3.4.1. *Let X and Y be Banach spaces with $X \subseteq Y$. The annihilator of X is the set $X^{\perp} = \{y^* \in Y^* \mid y^*(x) = 0, \forall x \in X\}$.*

Definition 3.4.2. *Let* X *be a real or complex Banach space*

*(a) A linear projection Ω is called an **M**-projection if $\|x\| = \max\{\|\Omega x\|, \|x - \Omega x\|\}$ for all $x \in X$. It is equivalently an L-projection if $\|x\| = \|\Omega x\| + \|x - \Omega x\|$ for all $x \in X$.*

(b) *A closed subspace of $X \subset Y$ is called an $M-$ summand if it is the range of M-projection. It is equivalently an $L-$ summand if it is the range of $L-$ projection.*

(c) *A closed subspace of $X \subset Y$ is called an* ***M****-ideal if $X^{\perp}$ is $L-$* summand *in Y^{*}.*

Remark 3.4.1. *Every Banach space X contains the trivial M-summands $\{0\}$ and X. All the other M-summands are nontrivial. The same remark applies to L-summands and M-ideals.*
There is an obvious duality between L-projection and M-projection: Ω is an L-projection on X if and only if Ω^{*} is an M-projection on X^{*}. Ω is an M-projection on X if and only if Ω^{*} is an L-projection on X^{*}

This remark yields the following characterization of M-projections which is useful in the sequel:
A projection $\Omega \in L(X)$ is an M-projection if and only if

$$\|\Omega x_1 + (Id - \Omega)x_2\| \leq \max\{\|x_1\|, \|x_2\|\} \text{ for all } x_1, x_2 \in X \tag{3.1}$$

In fact (3.1) means that the operator $(x_1, x_2) \mapsto \Omega x_1 + (Id - \Omega)x_2$ from $X \oplus_{\infty} X$ to X is contractive whence its adjoint $x^{*} \mapsto (\Omega^{*}x^{*}, (Id - \Omega)^{*}x^{*})$ from X^{*} to $X^{*} \oplus_1 X^{*}$ is contractive, where (X $\oplus_p Y$ denotes the direct sum of two Banach spaces, equipped with l^p- norm). This means that Ω^{*} is an L-projection and Ω must be an M-projection. We note that there is only one M-projection of Ω with $X = \mathcal{R}(\Omega)(= \ker(Id - \Omega))$ if X is an M-summand and only one L-projection Ω with $X = \mathcal{R}(\Omega)(= \ker(Id - \Omega))$ if X is an L-summand. Consequently, there is a uniquely determined closed subspace of $\hat{X}$ such that

$$Y = X \oplus_{\infty} \hat{X}, \text{ respectively}$$

$$Y = X \oplus_1 \hat{X}.$$

Then $\hat{X}$ is called the complementary $M-$ (Resp. $L-$)summand. The duality of $L-$ and

M-projections may now be expressed as

$$Y = X \oplus_\infty \hat{X} \quad \text{if and only if } Y^* = X^\perp \oplus_1 \hat{X}^\perp.$$

$$Y = X \oplus_1 \hat{X} \quad \text{if and only if } Y^* = X^\perp \oplus_\infty \hat{X}^\perp.$$

It follows that M-summands are M-ideal and that the M-ideals X is an M-summand if and only if the L-summand complementary to $X^\perp$ is weak * closed. It is noted that the fact that X and $\hat{X}$ are complementary to L-summands in X means geometrically that B_X, the closed unit ball of X, is the convex hull of B_X and $B_{\hat{X}}$.

Proposition 3.4.1. *(a) If Ω is an M-projection on Y and Q is a contractive projection on Y satisfying $\Omega(Y) = Q(Y)$ then $\Omega = Q$.*

(b) If Ω is an L-projection on Y and Q is a contractive projection on Y satisfying $\mathrm{Ker}(\Omega) = \ker(Q)$ *then $\Omega = Q$.*

Proof. We first prove (b). Our argument follows that for an $L-$summand X in Y, there is a given $y \in Y$, one and only one best approximant x_0 in X, that $\|y - x_0\| = \inf_{x\in X} \|y - x\|$ namely the image of y under the L-projection on X. Let $X = \ker\ (\Omega)$. For $y \in Y$ we have $y - \Omega y \in \ker(\Omega) = \ker Q$, hence $\|y - (y - Qy)\| = \|Qy\|$

$$
\begin{aligned}
&= \|Q(y - (y - \Omega y)\| \\
&\leq \|Q\|.\|\Omega y\| \\
&\leq \|(y - (y - \Omega y).\|
\end{aligned}
$$

This means that $y - Qy \in \ker Q = \ker(\Omega)$ is at least as good an approximant to y in X as $y - \Omega y$ which is the best approximation. From the uniqueness of the best approximant one deduces $Qy = \Omega y$, thus $\Omega = Q$ as claimed. (a) follows from (b) since $\ker(\Omega^*) = \mathcal{R}(\Omega)^\perp = \ker(Q^*)$. □

The following result shows that M-ideals are "Hahn-Banach smooth"

Proposition 3.4.2. *Let X be an M-ideal in Y. Then every $y^* \in X^*$ has a unique norm preserving extension to a function $y^* \in Y^*$.*

Proof. By assumption, $X^\perp$ is an L-summand so that there is a decomposition

$$Y^* = X^\perp \bigoplus X^*.$$

But $X^\#$ can be explicitly decomposed since there are canonical isometric isomorphism $X^* \cong Y^*/X^\perp \cong X^\#$ so that $X^\# = \{y^* \in Y^*, \|y^*\| = \||y^*|_X : y^* \cdot x \neq 0, x \in X\|\}$ and the result follows. $\square$

Remark 3.4.2. *The above proposition enables us to consider a subspace J^* of X^* in a new decomposition given by*

$$X^* = J^\perp \oplus_1 J^* \text{ where } J(X)\Delta X \ .$$

What follows in the sequel is an answer to the question whether an M-ideal in a Banach space induces an M-ideal in a subspace or a quotient space. We begin by availing the following general facts.

Lemma 3.4.1. *Let X and Z be closed subspaces in a Banach space Y.*

(a.) *$X + Z$ is closed in Y if and only if $X^\perp + Z^\perp$ is closed in Y^* if and only if $X^\perp + Z^\perp$ is weak * Closed in Y^*. In this case $X^\perp + Z^\perp = (X \cap Z)^\perp$ and*

$$(X + Z)/X \cong \mathrm{Z}/(X \cap Z), \left(X^\perp + Z^\perp\right)/Z^\perp \cong X^\perp/\left(X^\perp + Z^\perp\right).$$

(b.) *Suppose $X^\perp$ is the range of a projection Ω such that $\left(\Omega Z^\perp\right) \subset Z^\perp$. Then the assertion of (a) holds. If Ω is contractive we even have*

$$\left(X^\perp + Z^\perp\right)/X \cong Z/(X \cap Z)$$
$$\left(X^\perp + Z^\perp\right)/Z^\perp \cong X^\perp/\left(X^\perp + Z^\perp\right).$$

If $Id - \Omega$ is contractive we have $\quad (X+Z)/Z \cong X/(X\cap)$.

In the lemma that follows isometry is established

Lemma 3.4.2. *Let Ω be a projection in Y and let $Z \subset X$ be a closed subspace. We suppose $\Omega(Z) \subset Z$ so that $\Omega\ Z : Z \to Z, y \mapsto \Omega y$ and $\Omega/Y : Y/Z \to Y/Z, y + Z \mapsto \Omega y + Z$ are well- defined projections. We have $\mathcal{R}(\Omega/Z) = \mathcal{R}(\Omega) \cap Z$ and $\mathcal{R}(\Omega/Z) = (\mathcal{R}(\Omega) + Z)/Z$. Also $\mathcal{R}(\Omega/Z) \cong \mathcal{R}(\Omega)/(\mathcal{R}(\Omega) \cap Z)$ if Ω is contractive. Moreover Ω/Z and R/Z are $L-$ (resp $M-$) projections if Ω is.*

The researchers in[45] proved a sort of hereditary property of M-ideals for $K(X,Y)$. More specifically they proved the following results.

Theorem 3.4.1. *Suppose that X^{**} or Y^* has the Radon-Nikodym property and that $K(X,Y)$ is an M-ideal in $L(X,Y)$*

(a) If X^* has the bounded compact approximation property with adjoint operators and Z is a closed subspace of Y, then $K(X,Z)$ is an M-ideal in $L(X,Z)$.

(b) If Y^* has the bounded compact approximation property with adjoint operators and E is a closed subspace of X, then $K(X/E,Y)$ is an M-ideal in $L(X/E,Y)$.

Using suitable Hahn-Banach Extension operators corresponding to ideal projections and using Feder-Sapher representation of the dual space of certain space of compact operators, the following ideal properties of $K(X,Z)$ in $L(X,Z)$ for a closed subspace Z of Y are also evident.

Theorem 3.4.2. *[13] Let X and Y be Banach spaces such that X^{**} or Y^* has Radon-Nikodym property. If $K(X,Y)$ is an M-ideal in $L(X,Y)$, then for every closed subspace Z of $Y, K(X,Z)$ is an in M-ideal $L(X,Z)$.*

Since a reflexive Banach space has the Radon-Nikodym property, then if X is a reflexive Banach space and Z is a closed subspace of a Banach space Y. If $K(X,Y)$ is an M-ideal then $K(X,Z)$ is an M-ideal in $L(X,Z)$.

Theorem 3.4.3. *[13]: Let X and Y be Banach spaces. Suppose that $K(X,Y)$ is an M-ideal in $L(X,Y)$*

(a) If X^* has metric compact approximation with adjoint operators, then $K(X,F)$ is an M-ideal in $L(X,F)$ for a closed subspace F of Y.

(b) If Y^* has the compact approximation property then $K(X/E,Y)$ is an M-ideal in $L(X/E,Y)$ for every closed subspace E of X.

In [34] and [15], it is shown that a closed subspace X of a Banach space Y is an M-ideal in Y if and only if for every finite dimensional subspace G in Y and every $\epsilon > 0$, there exists a linear operator $U : G \to X$ such that $Ux = x$ for all $x \in G \cap X$ and

$$\|Ux + y - Uy\| \leq (1+\epsilon)\max(\|x\|, \|y\|) \text{ for all } x, y \in G.$$

We now give a generalization of notions of $M-$ ideals in the tensor product of Banach algebras and their extensions to quotient images. These characteristics lead us to the classes of ideals in the next section.

3.5 u-Ideals and h-Ideals

Suppose Y is a real or complex Banach space, a closed subspace X of Y is a summand if there exists a contractive projection of Y onto X. We further say that a closed subspace X of Y is u-summand if there is a subspace Z so that $X \oplus Z = Y$, and if $x \in X, z \in Z$ then $\|x+z\| = \|x-z\|$.

If Y is a complex Banach space we say that X is a h-summand with h-complement Z if $X \otimes Y = Z$ and if $x \in X, z \in Z$ and $|\lambda| = 1$ then $\|x + \lambda Z\| = \|x+z\|$. If X is u-summand then the induced projection $P : Y \to X$ with $P(Y) = X$ and $\ker P = Z$ satisfies $\|I - 2P\| = 1$. Likewise if X is an h-summand then $\|I - (1+\alpha)P\| = 1$ whenever $|\alpha| = 1$ which is equivalent to saying that P is Hermitian.

Next, we provide a lemma in which we show that the projection P defined above is unique.

Lemma 3.5.1. *Suppose X is a closed subspace of Y. Then the projection P of Y onto X satisfying $\|I-2P\|=1$ is unique.*

Proof. Suppose that we have two projections P and Q such that $\|I-2P\|=\|I-2Q\|=1$, then

$$\begin{aligned}(I-2P)(I-2Q)=&(I-2Q)-2P(I-2Q)\\ &=I-2Q-2P+4PQ.\end{aligned}$$

But $Q(Y)=X$, we have $(PQ)Y=P(QY)=Qy$ where $y\in Y$ and $Qy\in X$, therefore

$$(I-2P)(I-2Q)=I-2Q-2P+4Q.$$

Thus we have

$$\begin{gathered}((I-2P)(I-2Q))^2=(I+2(Q-P)(I+2(Q-P)))\\ =I+2(Q-P)+2((Q-P)(I+2(Q-P)))\\ =I+2(Q-P)+2(Q-P)+4(Q-P)^2\\ =[I+2(Q-P)]^2\\ =I+2.2(Q-P)\end{gathered}$$

$$\begin{aligned}((I-2P)(I-2Q))^3=(I+4(Q-P))(I+2(Q-P))\\ =I+2(Q-P)+4(Q-P)+8(Q-P)^2\\ =I+2.3(Q-P)\text{ etc.}\end{aligned}$$

In general, we have $((I-2P)(I-2Q))^n=I+2\cdot n(Q-P)$ and since

$\|I-2n(Q-P)\|=\|I-2n(P-Q)\|\geq\|I-2n\|\|P-Q\|\to\infty$ as $n\to\infty$ if $\|P-Q\|\neq 0$.

And $\|((I-2P)(I-2Q))^n\|\leq\|I-2P\|^n\|I-2Q\|=1$. We have a contradiction unless $P=Q$.

□

Next assuming that X is a u-ideal in Y. Let V be u-complement of $X^{\perp}$ in Y^*. Let P be a projection on Y^*, so that $\|I - 2P\| = 1$ and where the range $\mathcal{R}(\mathcal{P}) = \mathcal{V}$ and $\ker P = X^{\perp}$, then we have the following Lemma:

Lemma 3.5.2. *If X is a u-ideal in Y then X is a u-summand if and only if V is weak*-closed.*

Proof. If V is weak *-closed then P is weak *-continous and so $P = Q^*$ where $\|I - 2Q\| = 1$ and $Q(Y) = X$.

Conversely suppose X is u-summand and let Q be a projection onto X with $\|I - 2Q\| = 1$. Then $I - Q^*$ has range $X^{\perp}$ and so $I - Q^* = I - P$ has P is weak *-continuous. $\square$

Motivated by the above lemma we say that X is a strict u-ideal or as in the complex space a strict h-ideal if V is a norming subspace of Y^*. Refer to [15] for much literature in this direction.

Definition 3.5.1. *If X is an arbitrary Banach space and $x^{**} \in X^{**}$, we define its u-constant $k_u(x^{**})$ to be the infimum of all such a's such that we can write $x^{**} = \sum_{n=1}^{\infty} x_n$ in the weak *-topology with $x_n \in X$ and such that for any n and $\theta_k = \pm 1$ for $1 \leq k \leq n$ we have $\|\sum_{n=1}^{n} \theta_k x_k\| \leq a$. We set $k_u(x^{**}) = \infty$ if no such a exists.*

Definition 3.5.2. *Let $Ba(X)$ be the collection of $x^{**} \in X$ such that there is a sequence (x_n) in X with $\lim x_n = x^{**}$. Then X has property (u) if $x^{**} \in Ba(X)$ has $k_u(x^{**}) \leq \infty$. Then by the closed graph theory, there exists some c so that $k_u(x^{**}) \leq c\|x^{**}\|$ for all $x^{**} \in Ba(X)$. We denote the least of such constants by $k_u(X)$.*

Definition 3.5.3. *If X is a complex Banach space, we define $k_h(x^{**})$ to be the infimum of all a's such that $x^{**} = \sum_{n=1}^{\infty} x_n$ and for any n, and any $|\theta_k| = 1$ for $1 \leq k \leq n$, we have $\|\sum_{n=1}^{n} \theta_k x_k\| \leq a$. Clearly $k_u(x^{**}) \leq k_h(x^{**}) \leq 2k_u(x^{**})$.*

Definition 3.5.4. *If X has property (u) we define $k_h(X)$ to be the least constant c so that $k_h(x^{**}) \leq c\|x^{**}\|$ for $x^* \in Ba(X)$. Thus $k_u(X) \leq k_h(X) \leq 2k_u(X)$.*

We are now in position to show the lemma that follows:

Lemma 3.5.3. *Suppose that X is a u-ideal (respectively an h-ideal) in Y. Suppose that $y \in Y$ and $\epsilon > 0$. Let A be a convex subset of X such that Ty is in the weak *-closure of A and that B is any compact subset of X. Then there exists $x \in A$ such that $\|y - (1+\lambda)x - \lambda Z\| < \|y+z\| + \varepsilon$ whenever $-1 \leq \lambda \leq 1$ (respectively $|\lambda| \leq 1$) and $Z \in B$.*

Proof. We may assume that $0 \in B$. Let $M = \max\{\|x\| : Z \in B\}$ and pick $0 < \delta < 1$ so that $(M + 4 + 2\|y\|)\delta < \varepsilon$. Let $\{\lambda_1, \ldots, \ldots, \lambda_m\}$ be a δ-net for the closed unit disk, which we suppose includes zero and let $\{z_1, \ldots, ..z_n\}$ be a net for B, also including zero. For any subset J of $\Omega = [m] \cap [n]$, define H_J to be the set of $x \in A$ such that

$$H_J = \|y - (1+\lambda_j)\, x - \lambda_j Z_k\| < \|y + Z_k\| + \delta$$

whenever $(j,k) \in J$. Thus $H_\theta = A$. We proceed to show that H_Ω is none empty. Pick any $(j,k) = \Phi$ where $K = J \cup \{(i,j)\}$. However $A^1 = W \cap H_J$ is convex and Ty is in its weak *-closure. Thus $T\,(y + Z_k)$ is in the weak *-closure of $A^1 + Z_k$. There exists $x \in W \cap H_J$ such that $\|y + Z_k - (1+\lambda_j)\,(x + Z_k)\| < \|y + Z_k\| + \delta$. On reorganizing this implies that $x \in W \cap H_K$, which is contrary to the assumption that $(j,k) = \Phi$. It then follows that $H_\Omega \neq \Phi$.

Next pick any $x \in H_\Omega$, then if $x \in B$ and $|\lambda| \leq Y$, we may find $(i,j) \in \Omega$ such that $|\lambda - \lambda_j| \leq \delta$ and $|Z - Z_j| \leq \delta$. Thus

$$\|\mathrm{y} - (1+\lambda)x - \lambda Z\| \leq \|y - (1+\lambda_j)\, x - \lambda_j Z_k\| + \delta\,(1 + \|x\| + \|Z_k\|)$$

$$\leq \|y + Z_k\| + \delta\,(1 + \|x\| + \|Z_k\|)$$

$$\leq \|y + Z\| + \delta\,(3 + \|x\| + \|Z_k\|)\,.$$

Zero appears in both δ-net so we have $\|y-x\| \leq \|y\|+\delta$ thus $\|x\| \leq 2\|y\|+1$. hence

$$\delta\left(3+\|x\|+\|Z_k\|\right) \leq (M+4+2\|y\|)\delta < \varepsilon$$

$$\delta\left(3+\|x\|+\|Z_k\|\right) < \varepsilon.$$

$\square$

In the following proposition we characterize u-ideals and h-ideals.

Proposition 3.5.1. *Let Y be a Banach space and let X be a closed subspace of Y. For X be a $u-$ ideal (respectively h-ideal) in Y, it is necessary and sufficient that for every finite-dimensional subspace F of Y and every $\varepsilon > 0$, there is a linear map $L : F \to X$ such that $Lx = x$ for $x \in F \cap X$ and $\|f - 2Lf\| \leq (1+\varepsilon)\|f\|$ for every $f \in F$ (respectively $\|f-(1+\lambda)Lf\| \leq (1+\varepsilon)\|f\|$ for every $f \in F$ and for every λ such that*

$$|\lambda| = 1).$$

Proof. We prove the result in the u-ideal case since for the case of h-ideal the result holds with slight modifications.

Suppose that X is a u-ideal in Y and that F is a finite dimensional subspace of Y. Then we claim that $L(F,X)$ is a u-ideal in $L(F,Y)$. In fact $L(F,X)^*$ can be identified with $F \otimes_\Pi Y^*$ and so we can induce a projection P on it by $P\left(f \otimes_\Pi y^*\right) = f \otimes Py^*$. It is clear that $\|1-2P\| = 1$ and that $\ker P = L(F,X)^\perp$. Further P induces a map $T : L(F,Y) \to L\left(F, X^{**}\right)$ in the usual way so that for any $\Phi \in L(F,X)^*$ we have $T(L)(\Phi) = P(\Phi)(L)$. Let $J : F \to Y$ be the identity map and $\mathcal{A}$ be the collection of all $\mathcal{L} \in L(F,X)$ such that $\mathcal{L}x = x$ for all $x \in P \cap X$. Suppose $T(J)$ is not in the weak *-closure of $\mathcal{A}$, there exists $\Phi \in P \cap X$. Suppose $T(J)$ is not in the weak *-closure of $\mathcal{A}$, then there exists $\Phi \in L(F,X)^*$ so that $\sup\limits_{L \in \mathcal{A}} \mathcal{R}\Phi(L) = \alpha < \mathcal{R}T(J)(\Phi)$. Clearly $\Phi(S) = 0$ if $S = 0$ on $F \cap X$. It follows that we can write $\Phi = \sum_{j=1}^m f_j \otimes {x_j}^*$ where $\{f_j\}$ is abasis for

$F \cap X$, and $x_j{}^* \in X^*$. Let $y_j{}^*$ be extensions of $x_j{}^*$ to Y^*, then

$$\langle \Phi, T(J) \rangle = \left\langle J, P\left(\sum_{j=1}^{m} f_j \otimes y_j^*\right)\right\rangle = \sum_{j=1}^{m} \langle J, f_j \otimes P y_j{}^* \rangle = \sum_{j=1}^{m} \langle f_j, P y_j{}^* \rangle = \sum_{j=1}^{m} x_j{}^* \left(f_j\right).$$

Now letting S be any projection of F onto $F \cap X$ we find that $\Phi(S) = T(J)(\Phi)$. Thus $T(J)$ is in the weak *-closure of $\mathcal{A}$ and so there exists $\mathcal{L} \in \mathcal{A}$ so that

$$\|J - 2\mathcal{L}\| < 1 + \varepsilon.$$

Conversely, suppose for every finite-dimensional F and $\varepsilon > 0$, there exists $\mathcal{L} = \mathcal{L}_{F,\varepsilon} : F \to X$ so that $\mathcal{L}x = x$ for $x \in F \cap X$ and $\|f - 2\mathcal{L}f\| \leq (1+\varepsilon)\|f\|$ for $f \in F$. We regard the collection of all pairs (F, ε) as a directed set in the obvious way. Extending $L_{F,\varepsilon}$ to a nonlinear operator $L_{F,\varepsilon} : Y \to X$ by setting $L_{F,\varepsilon}(x) = 0$ for $x \notin F$. By compactness argument, we find a subnet (L_d) of $L_{F,\varepsilon}$ so that for every $y^* \in Y^*, y \in Y, \lim_j y^* \left(L_d y\right) = h\left(y^*, y\right)$ exists. Then $y \to h\left(y^*, y\right)$ is linear and bounded and so we can define $Py^* \in Y^*$ by $\langle y, Py^* \rangle = h\left(y^* y\right)$. Further P is linear, $\ker P = X^{\perp}$ and $\|I - 2P\| = 1$, the ideal property. □

3.6 Embedments in the Biduals

We consider the u and $h-$ ideals of X embedded in the biduals X^{**} where X is a Banach Algebra.

Definition 3.6.1. *We will say that X is a u-ideal (respectively an h-ideal) if X is a u-ideal (respectively an h-ideal) in X^{**} for the canonical embedding.*

Proposition 3.6.1. *A Banach space X is a u-ideal (respectively an h-ideal) if given a finite-dimensional subspace F of X^{**} and $\varepsilon > 0$ there exists a linear map $L : F \to X$ so that $Lx = x$ for $x \in X \cap F$ and $\|f - 2Lf\| \leq (1+\varepsilon)\|f\|$ for $f \in F$ (respectively $\|f - (1+\lambda)Lf\| \leq (1+\varepsilon)\|f\|$ for $|\lambda| = 1$ and $f \in F$).*

Remark 3.6.1. *Suppose X is a Banach space so that for every $\varepsilon > 0$ there is a u-ideal (respectively an h-ideal) Y so that X is $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of Y. Then X is a u-ideal (respectively an h-ideal).*

Next, we turn to general Theory of h-ideals. We choose not to put any restrictions. We shall exploit the Hermitian operators on the general Banach space. The following general result will be the first in this direction.

Lemma 3.6.1. *Let X be an arbitrary complex Banach space. Let $H : X^{**} \to X^{**}$ be a Hermitian operator such that $Hx = 0$ for $x \in X$ with $\|x^{**}\| = 1$, then there exists a sequence (x_d) in X such that $\lim_d x_d = Tx^{**}$ weak * and for every $|\lambda| = 1$,* $\limsup_d \|x^{**} - (1+\lambda)x_d\| \leq 1$.

Proof. We assume first that X is an h - ideal. Next we define for all real t, the operator $\exp(itP)$ invertible and isometric on X^{**} which satisfies $\exp(itP)x = x$ for all $x \in X$.
We need to show that there exists a projection T onto $B_a(X)$. Now $Tx^{**} = x^{**}$ for $x^* \in B_a(X)$, we claim that if $x^{**} \in B_a(X)$ is such that the ker x^{**} is norming then $x^{**} = 0$. This follows from the fact that $K_h(x^{**}) = \|x^{**}\|$. Now if X is separable there is a separable norming subspace M in X^*. If $x^{**} \in S_{X^{**}}$ and $\varepsilon > 0$ we can find $\chi \in B_a(X)$ with $K_h(\chi) < 1+\varepsilon$ and such that $\chi(f) = Tx^{**}(f)$ for $f \in M$ and $\chi_1(x^*) = Tx^{**}(x^*)$. This follows by applying the same argument to the span of M and x^*. Thus $\chi_1 - \chi \in B_a(X)$ and $M \subset \mathrm{Ker}(\chi_1 - \chi)$. It also follows that $\chi_1 = \chi$ and hence that $T\chi^{**}(x^*) = \chi(x^*)$ for all $x^* \in X^*$. Thus $Tx^{**} = \chi \in B_a(X)$. Hence T is a Hermitian projection onto $B_a(X)$.
Conversely, We define $P : X^{***} \to X^{***}$ by $P = T^*\pi$. It is clear that πT^* is a projection of X^{***} onto X^* and so P is a projection whose kernel is $X^{\perp}$. So, for each $x^{**} \in S_{X^*}$ there exists a net (x_d) with $\lim_{im} x_d = Tx^{**}$ weak * and so $|\lambda| = 1$ then $\limsup \|x^{**} - (1+\lambda)x_d\| \leq 1$. Thus, it follows that $\|1-(1+\lambda)P\| = 1$ if $|\lambda| = 1$ and this shows that $\|\exp(itP)\| = 1$ for all real t, that is, P is Hermitian. □

The Theorem that follows characterizes h-ideals. First, we provide the Lemmata that shall be used in its proof.

Lemma 3.6.2. *T is Hermitian on X^{**} and $T(x) = x$ for $x \in X$.*

Lemma 3.6.3. *Let X be arbitrary real or complex Banach space and suppose $x^{**} \in S_{X^{**}}$ satisfies $k_u(x^{**}) < 2$. Then ker x^{**} cannot be a norming subspace of X^*.*

Proof. Let $x^{**} = \sum x_n$ weak * where if $\theta_k = \mp 1$ then, $\|\sum_{k=1}^{n} \theta_k x_k\| \leq 2 - \delta$ for some $\delta > 0$. If $S_n = \sum_{k=1}^{n} x_k$ then $\|x^{**} - 2S_n\| \leq 2 - \delta$. Then there is a sequence of convex combinations t_n that converges weak * to x^{**} and $\lim \|t_n\| = 1$. Thus $\|x^{**} - 2t_n\| \leq 2 - \delta$, which leads to the fact that if $x^* \in \ker x^{**}$ and $\|x^*\| = 1$ then $|x^*(t_n)| \leq 1 - \delta/2$. Thus $ker x^{**}$ cannot be norming.

□

Theorem 3.6.1. *Let X be a h-ideal. Then the following are equivalent*

1. *X is a strict h-ideal*
2. *X^* is an h-ideal*
3. *Every separable subspace of X has a separable dual.*
4. *$\|I - \lambda\pi\| \leq 1$ if $|\lambda - 1| \leq 1$*
5. *X contains no copy of l_1*

Next, we consider how we can identify h-ideals and this we do by considering first subspace.

Theorem 3.6.2. *Let X be a separable h - ideal and T be the induced Hermitian projection of X^{**} onto $Ba(X)$. If Z is a subspace of X such that $Z^{\perp\perp}$ is T-invariant then Z is an h-ideal.*

Proof. $Z^{\perp\perp}$ can be identified with Z^{**} and T restricted to an Hermitian projection on Z^{**} whose range includes Z. If $z^{**} \in Z^{**}$ with $\|z^{**}\| = 1$ then Tz^{**} is in the weak *-closure of B_z and so there is a net (z_d) in Z with $\lim z_d = TZ^{**}$ week* and $\limsup_d \|Z^{**} - (1 + \lambda) z_d\| \leq 1$ whenever $|\lambda| = 1$ and the result follows. □

Definition 3.6.2. *We say that a separable $h-$ ideal is non degenerate if whenever $\chi \in \ker T$ and $x^{**} \in Ba(X)$, then $\|\chi + x^{**}\| = \|\chi\|$ implies $x^{**} = 0$.*

Theorem 3.6.3. *Let X be separable non degenerate h - ideal.Then a closed subspace Z of X is an h-ideal if and only if $Z^{\perp\perp}$ is T-invariant.*

Proof. Consider a h - ideal Z and $T_z : Z^{**} \to Ba(X)$ be the associated Hermitian projection. Then $l_1 - \text{sum}\, X \oplus_1 Z$ is also an h - ideal and the associated projection of $X^{**} \oplus_1 Z^{**}$ onto $Ba(X) \oplus_1 Ba(Z)$ is given by $T \oplus T_Z$. Suppose $\chi \in Z^{**}$ satifies $T_z\chi = 0$ and $\|\chi\| = 1$. Identifying Z^{**} with $Z^{\perp\perp} \subset X^{**}$ in the natural way to consider χ in X^{**}. Then $T\chi \in B_a(X)$ and so there is a sequence (u_n) in X converging weakly to $T\chi$. Let $\xi = \chi - T\chi$ and $C_n \subset X \oplus_1 Z$ be the set of all (x, z) such that $z - x \in C_n \{u_k : k \geq n\}$, then (ξ, χ) is the weak * - closure by each C_n. If $\delta > 0$ then (ξ, χ) is also in weak *-closure of $A_n = \{(x, z) \in C_n : \|x\| \leq (1+\ \delta)\|\xi\|, \|Z\| \leq 1 + \delta\}$. In fact if $B = \{(x, z) : \|x\| \leq \|\xi\|, \|Z\| \leq 1\}$, then for any weak *-neighbourhood W of $(\xi, \chi), 0$ is in the weak-closure of $(W \cap C_n) - B$ and hence also in norm - closure. Hence $0 \in (W \cap C_n) - (1 + \delta)B$, whence $W \cap C_n \cap (1 + \delta)B$ is nonempty. It follows that we can pick $(x_n, z_n) \in A_n$ so that for all scalars $\alpha_1, \ldots, \ldots, \alpha_n$ and all $n \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{n} \alpha_k x_k\right\| + \left\|\sum_{k=1}^{n} \alpha_k z_k\right\| \geq (1 - \delta)(1 + \|\xi\|) \sum_{k=1}^{n} |\alpha_k| .$$

By construction $\lim (z_n - x_n) = T\chi$ weak * and so $\lim (z_{2n} - z_{2n+1} - x_{2n} + x_{2n+1}) = 0$ weakly. Thus, there exists $n \in \mathbb{N}, \beta_k \geq 0$ such that $\sum \beta_k = 1$ and

$$\left\|\sum_{k=1}^{n} \beta_k (z_{2k} - z_{2k+1}) - \sum_{k=1}^{n} \beta_k (x_{2k} - x_{2k+1})\right\| \leq \delta.$$

It follows that

$$\left\|\sum_{k=1}^{n} \beta_k (z_{2k} - z_{2k+1})\right\| \leq 2(1 + \delta)\|\xi\| + \delta$$

and hence that that

$$\left\|\sum_{k=1}^{n}\beta_k\left(z_{2k}-z_{2k+1}\right)\right\|+\left\|\sum_{k=1}^{n}\beta_k\left(x_{2k}-x_{2k+1}\right)\right\|\leq 4(1+\delta)\|\xi\|+\delta$$

Hence

$$2(1-\delta)(1+\|\xi\|)\leq 4(1+\delta)\|\xi\|+\delta$$

and as $\delta>0$ is arbitrary this implies as $\delta\to 0$ we have $2(1+\|\xi\|)\leq 4\|\xi\|$, and $1\leq\|\xi\|$. Hence $\|\xi\|=1$ and $\xi+T\chi=T\chi$ and X is non degenerate $h-ideal$ and this in turn implies $T\chi=0$. It follows immediately that $T_z\chi=\mathrm{T}\chi$ for any $\chi\in Z^{\perp\perp}$ and so $Z^{\perp\perp}$ is T-invariant. □

3.7 u-ideals

For the $u-$ideals, we first consider the case when X contains no subspace isomorphic to l_1.

Proposition 3.7.1. *Let X be a Banach space containing no copy of l_1 which is a u-ideal. Then V is weak *-dense in X^{***}.*

Proposition 3.7.2. *Let X be a Banach space containing no copy of l_1. Suppose P is a projection on X^{***} such that* $\ker P=X^{\perp}$ *and $\|P\|=1$. Let $V=P\left(X^{***}\right)$. Then $V\cap X^*$ is norming for X.*

Proof. We consider the associated map T. It is clear from the definition that if $x^*\in X^*$ and $T^*x^*=x^*$ then $x^*\in V$. Now for each $\chi\in X^{**}$ consider the set

$$E_\chi=\left\{x^*\in X^*:T\chi\left(x^*\right)=\chi\left(x^*\right)\right\}.$$

Now χ is of the first Baire class on (B_{X^*},w^*) and therefore the set $C_*(\chi)$ of points of continuity is a dense G_d-set. Assume $x^*\in C_*(\chi)$. Let $v=Px^*$. Then there is a net (x_a^*) in B_{X^*} converging in the weak * topology of X^{***} to v. However, $(v-x^*)\in X^{\perp}$ so that ${x^*}_d$

converges for $\sigma(X^*, X)$ to x^*. Thus $v(\chi) = \lim_d \chi(x_d^*) = \chi(x_d^*)$ so that $Px^*(\chi) = \chi(x^*)$. Now $T\chi(x^*) = Px^*(\chi)$ so, we conclude that $x^* \in E_{\chi^*}$. Hence E_χ is norming and further this implies that $H = \cap_{X \in X} \ldots E_X$ is also norming provided that $H \subset V \cap X^*$. $\square$

In the Theorem that follows we look at a more general case of separable $u-$ideal and prove the result

Theorem 3.7.1. *Let X be a separable u-ideal such that $k_u(X) \leq 2$. Then $k_u(X) = 1$ and $Ba(X)$ is a u-summand in X^{**}.*

Proof. If $x^{***} \in Ba(X)$ is such that $\ker x^{***}$ is norming then $x^{***} = 0$. Suppose $x^{**} \in X^{**}$. Let M be a separable norming subspace of X^*. Suppose $\varepsilon > 0$, then there exists $\chi \in X^{**}$ with $k_u(\chi) \leq \|x^{**}\| + \varepsilon$ so that $\chi(f) = Tx^{**}(f)$ for all $f \in M$. This leads to the conclusion that $\chi = Tx^{**}$. Thus T maps X^{**} into $B_a(X)$. Next if $x^{**} \in B_a(X)$, then the set of $x^* \in B_{X^*}$ such that $x^{**}(x^*) = Px^{**}(x^*)$ contains a weak $^*-$ dense G_d-subset. Hence $x^{**} - Tx^{**}$ vanishes on a norming subspace of X^* and is in $B_a(X)$. Hence $Tx^{**} = x^{**}$. Thus T is a projection of X^{**} onto $B_a(X)$ and of course $\|I = 2T\| = 1$. It further follows that if $x^{**} \in B_a(X)$ then $k_u(x^{**}) = \|x^{**}\|$. $\square$

Now, let X be a closed subspace of a Banach space Y and let i_X be the natural embedding $i_X : X \to Y$. If P is a norm one projection on Y^* with $\ker P = X^\perp$ we may define a norm one operator $T : Y \to X^{**}$ by letting $\langle i_X{}^* y^*, T(y)\rangle = \langle y, P(y^*)\rangle$ for all $y \in Y$ and $y^* \in Y^*$. Then $T(x) = x$ for all $x \in X$ and if $(I - 2P)$ is an isometry then $\|y - 2i_{x^{**}}T(y)\| = \|y\|$ for all $y \in Y$. Further more if we let $V = P(Y^*)$, then X being a u-ideal in Y means that $Y^* = V \oplus X^\perp$ and $\|v + \eta\| = \|v - \eta\|$ for all $v \in V$ and $\eta \in X^\perp$. Consequently, we have the following results which agrees with the notions in [14].

Lemma 3.7.1. *Let X be a closed subspace of a Banach space Y. If X is a u-ideal in Y then for every $\varepsilon > 0$, $y \in Y$ and $x \in X$ there is $x_0 \in X$ such that $\|y + x - 2x_0\| = \|y - x\| + \varepsilon$.*

Theorem 3.7.2. *Let X be a closed subspace of a Banach space Y. The following statements are equivalent.*

(a) X is a u-ideal in Y.

(b) X is a u-ideal in Z for every subspace Z of Y with $\dim Z/X < \infty$.

(c) X is a u-ideal in Z for every subspace Z of Y with $\dim Z/X \leq 2$.

3.8 Co-Dimension One

In this section, we emphasize the local properties of ideals that depict the global properties. We further show that the ball intersection property is inherited by quotients and prove general results about centers of symmetry in the compact convex sets.

Let X be a closed subspace of a Banach spaceY. For $y \in Y \backslash X$ we can define the set of best approximants $P_y = \left\{x^{**} \in X^{**} : \|y - i_X{}^{**}\left(x^{**}\right)\| = d\left(y, X^{\perp\perp}\right)\right\}.P_y$ is a non-empty weak $^{*}-$ compact convex subset of X^{**}. We give a number of lemmas crucial in one of our main results.

Lemma 3.8.1. *Let X be a closed subspace of a Banach space Y.Then, $I - P$ has norm one if and only if $T(y) \in P_y$ for all $y \in Y$.*

Proof. If $T(y) \in P_y$ then,

$$\|I - P\| = \sup_{y^* \in B_{Y^*} \dots} \sup_{y \in B_Y} |\langle y, y^* - Py^* \rangle| = \sup_{y^* \in B_{Y^*}} \sup_{y \in B_Y} |\langle y^*, y - i_X{}^{**}Ty \rangle|$$
$$\leq \sup_{y \in B_Y} \|i_X{}^{**}Ty - y\| \leq \sup_{y \in B_Y} d\left(y, X^{\perp\perp}\right) \leq \sup_{y \in B_Y} \|y - 0\| \leq 1.$$

So we have $\|I - P\| \leq 1$.

Conversely if $\|I - P\| = 1$ then,

$$\|y - i_x{}^{**}Ty\| = \|(y - x) - i_x{}^{**}T(y - x)\| \leq \|y - x\|.$$

So that $T(y) \in P_y$. □

Remark 3.8.1. *If $\|I-P\|=1$ then both $I-P$ and P have norm one. $T(y)\in P_y$ is the Center of symmetry.*

Definition 3.8.1. *An element C in a convex set J is a center of symmetry if $2C-x\in J$ for all $x\in J$. C is a centre of symmetry if and only if $K-C$ is symmetric about the origin. This center of symmetry is unique.*

Lemma 3.8.2. *Let X be a closed subspace of a Banach space Y. If X is a u-ideal in Y then $T(y)$ is a center of symmetry in P_y for all $y\in Y$.*

Proof. Let $x^{**}\in P_y$. Since I^*-2P^* is an isometry we have $d\left(y,X^{\perp\perp}\right)=\|y-{i_X}^{**}\left(x^{**}\right)\|.$

$$\begin{aligned}&=\|y-i_x^{**}\left(x^{**}\right)-2P^*\left(y-i_x^{**}\left(x^{**}\right)\right\|\\&=\|y+i_x^{**}\left(x^{**}-2T(y)\right)\|.\end{aligned}$$

So that $2T(y)-x^{**}\in P_y$. □

The next result emphasizes the local properties of ideals. They are aimed at depicting the global properties of the said ideals.

Proposition 3.8.1. *Let X be a closed subspace of a Banach space Z such that $Z/X<\infty$, then X is a u-ideal in Z if and only if for every subspace $W\subseteq X$ of finite co-dimension X/W is a u-ideal in Z/W.*

Proof. Let X be a u-ideal in Z and let $W\subseteq X$ be a finite co-dimensional subspace. Let $T:Z\to X^{**}$ with $T(x)=x$ for all $x\in X$ and $\|z-2T(z)\|=\|z\|$ for all $z\in Z$. Let $Q_W:Z\to Z/W$ be the quotient map. Define $T_W:Z/W\to X/W=(X/W)^{**}$ by

$$T_W\left(Q_W(Z)\right)=Q_W^{**}\left(i_X^{**}(T(z))\right)$$

Which is well defined since $T_W(0)={Q_W}^{**}\left({i_X}^{**}(T(W))\right)=Q_W(W)=0$. We have

$$\sup_{W_W(z)\in B_{z/W}}\|Q_W(z)-2T_W\left(Q_W(z)\right)\|=\sup_{z\in B_z}\|{Q_W}^{**}(z)-2{Q_W}^{**}\left({i_X}^{**}(T(z))\right)\|\leq 1.$$

And for $Q_W(x) \in X/W$

$$T_W\left(Q_W(x)\right) = {Q_W}^{**}\left({i_X}^{**}(T(x))\right) = {Q_W}^{**}\left({i_X}^{**}(x)\right) = Q_W(x).$$

By finite dimensionality of X/W and weak *-continuity of both ${Q_W}^{**}$ and ${i_X}^{**}$, we get that T_W is contained in X/W. Thus, X/W is a u-ideal in Z/W.

Conversely, let C_X denote the set of all finite-co-dimensional subspaces in X and suppose X/W is a u-ideal in Z/W for all $W \in C_X$. Let $W \in C_X$ and $Q_W : Z \to Z/W$. We have $\dim X/W < \infty$ and $\dim Z/W < \infty$ and as above we consider X/W as a subspace of Z/W and identify $Q_W(X)$ with X/W. We can therefore identify $(X/W)^\perp$ with $X^\perp \subseteq (Z/W)^* = (W)^\perp$ in Z^*. By assumption there is a projection $P_W : W^\perp \to W^\perp$ with $\operatorname{Ker} P_w = X^\perp$. Let $\mathcal{U}$ be an ultrafilter refining, the reverse order filter on C_X. Define $P : Z^* \to Z^*$ by

$$P\left(z^*\right) = \omega^* - \lim_{u} P_W\left(z^*\right).$$

Then

$$\|z^* - 2Pz^*\| \leq \lim_{u} \|z^* - 2P_W z^*\| \leq \|z^*\|$$

and $\ker P = X^\perp$ since $z^* \in Z^* \backslash X^\perp$ is in $W^\perp \backslash X^\perp$ eventually.

$\square$

The lemma that follows shows that the ball intersection property is inherited by quotients.

Lemma 3.8.3. *Let X be a closed subspace of a Banach space Y and let $y \in Y \backslash X$. Assume that $X \cap_{i=1}^{3} B_Y\left(y + x_i, \|y - x_i\| + \varepsilon\right) \neq \emptyset, \varepsilon > 0$ for every collection of three points $(x_i)_{i=1}^{3} \subset X$.If W is a finite co-dimensional subspace of X then X/W has the property*

$$X \cap_{i=1}^{3} B_Y\left(y + x_i, \|y - x_i\| + \varepsilon\right) \neq \emptyset, \varepsilon > 0 \textit{ in } Y/W \textit{ with respect to } y + W.$$

Proof. Let $Q_W : Y \to Y/W$ denote the quotient mapping. We consider X/W as a subspace of Y/W. Let $\varepsilon > 0$ and $(u_i)_{i=1}^3 \subset X/W$. Choose $x_i \in X$ and such that $Q_W(x_i) = u_i$ for $i = 1, 2, 3$. Since $W \subset X$, we may assume that:

$$\|y - x_i\| < \|Q_W(y - x_i)\| + \varepsilon = \|Q_W(y) - u_i\| + \varepsilon.$$

Choose $x \in X \cap_{i=1}^3 B_Y(y + x_i, \|y - x_i\| + \varepsilon)$. Then

$$Q_W(x) \in X/W \cap_{i=1}^3 B(Q_W(y) + u_i, \|Q_W(y) - u_i\| + \varepsilon)$$

as desired. □

We state and prove one of our main results in the sequel.

Theorem 3.8.1. *Let X be a closed subspace of a Banach space Y and let $y \in Y \backslash X$ and $Z = \operatorname{span}(X, \{y\})$. Then the following statements are equivalent.*

(i) X is a u-ideal in Z

(ii) $X^{\perp\perp} \cap (\cap_{x \in X} B_{z^{**}}(y + x, \|y - x\|)) \neq \phi$.

(iii) $X \cap (\cap_{i=1}^n B_z(y + x_i, \|y - x_i\| + \varepsilon)) \neq \phi$ for every finite collection $(x_i)_{i=1}^n \subset X$ and $\varepsilon > 0$.

(iv) $X \cap (\cap_{i=1}^3 B_z(y + x_i, \|y - x_i\| + \varepsilon)) \neq \phi$ for every finite collection of three points $(x_i)_{i=1}^3 \subset X$ and $\varepsilon > 0$.

Proof. *(i)* $\Rightarrow$ *(ii)* : Let $T : Z \to X^{**}$ be the operator associated with u-ideal projection. For $x \in X$ we have

$$\|y - x\| = \|y - x - 2{i_x}^{**}T(y - x)\| = \|y + x - 2{i_x}^{**}T(y)\|.$$

This means that $2{i_x}^{**}T(y) \in B(y + x, \|y - x\|)$.

(ii) $\Rightarrow$ (iii). Let $x^{***} \in X^{\perp\perp} \cap \cap_{x \in X} B_{z^{**}}(y + x, \|y - x\|) \neq \emptyset$. We use the principle of local reflexivity with $E = \operatorname{span}\left(i_X^{**}\left(x^{**}\right), y, x_1, x_2, \ldots, x_n\right) \subset Z^{**}$ and $= X^{\perp} \subset Z^*$.

$(iii) \Rightarrow (iv)$. Since $i = 1, 2, 3$ gives the three points of (iv).

(iv) $\Rightarrow$ (i). Let X be a finite co-dimensional subspace, then it is sufficient to show that X/W is a u-ideal in the quotient X/W which has property (iv) in Z/W. This reduces the problem to one which is finite-dimensional. Let $r_y = d\left(Q_W(y), X/W\right)$ and let $Q_W : Z \to Z/W$ be the quotient mapping. By finite dimensionality there is at least one exposed point $e_0 \in P_{Q_W(y)}$ with exposing functional $e^* \in (X/W)^*$. Let $M = e^*\left(e_0\right) = \max_{PQ_W(y)} e^*(e)$ and find $e_1 \in P_{Q_W(y)}$ such that

$$m = e^*\left(e_1\right) = \min_{P_{Q_W(y)}} e^*(e). \text{ Choose}$$

$$2c \in X/W \cap B\left(Q_W(y) + e_0, r_y\right) \cap B\left(Q_W(y) + e_1, r_y\right).$$

We get $2c - e_1 \in P_y$ for $i = 0, 1$ and

$$M \geq e^*\left(2c - e_1\right) = 2e^*(c) - m$$

$$m \leq e^*\left(2c - e_0\right) = 2e^*(c) - M$$

So that $e^*(c) = \frac{M+m}{2}$ and $M = e^*\left(2c - e_1\right)$. Since e_0 is exposed by e^* we get $c = \frac{e_0+e_1}{2}$ and c is also unique. By assumption we have

$$\{2c\} = \bigcap_{i=0,1} B\left(Q_W(y) + e_i, r_y\right) \cap B\left(Q_W(y) + u, \|Q_W(y) - u\|\right)$$

for all $u \in X/W$ and this means that

$$\|Q_W(y) + u - 2c\| \leq \|Q_W(y) - u\| \, \forall \, u \in X/W.$$

So by the hereditary properties between spaces and their quotients, $X/W \Delta_u Z/W$ in the algebraic structures as required. $\square$

In the proposition that follows we prove general results about centres of symmetry in the compact convex sets.

Proposition 3.8.2. *Let K be a convex compact set in a locally convex Hausdorff vector space X. Given for every finite-dimensional Banach space Y and every continuous linear operator $T : X \to Y$, $T(K)$ has a center of symmetry then, K has a center of symmetry.*

Proof. For Y a finite-dimensional Banach space and $T : X \to Y$ a continuous linear operator, define $C_{Y,T} = \{x \in K : T(x) \text{ is a center of symmetry in } T(K)\}$. Every $C_{Y,T}$ is non-empty, convex and compact. By taking a finite sum of finite dimensional Banach spaces, we see that the family $C_{Y,T}$ has the finite intersection property. By compactness, there exists $c \in \bigcap_{Y,T} C_{Y,T}$ which is a center of symmetry in K. Indeed, let $x \in K$ and assume $2c - x \notin K$. Then there exists an $x^* \in X^*$ such that $x^*(2c - x) > \sup_{u \in K} x^*(u)$, but this contradicts $c \in C_{\mathbb{R},x^*}$. □

3.9 Co-Dimension Two

We make the following simple observation in form of a lemma

Lemma 3.9.1. *Let Y be a Banach space and suppose X and Z are subspaces of Y such that $X \subseteq Z$ and denote natural embeddings by $i_X : X \to Y, i_Z : Z \to Y$ and $j : X \to Z$. For $y \in Z$ and $x^{**} \in X^{**}$ we have:*

$$\|y - i_X^{**}(x^{**})\| = \|y - j^{**}(x^{**})\|.$$

In particular, the set $P_y \subset X^{**}$ is the same whether it is defined relative to Y or Z. Moreover, $d\left(y, X^{\perp\perp}\right) = d(y, X)$.

Proof. We have $i_X = i_Z j$. Let $x^{**} \in X^{**}, y \in Z \backslash X$ and $y^* \in Y^*$. We have

$$\langle y - {i_X}^{**}(x^{**}), y^* \rangle = \langle i_z(y) - i_z^{**} j^{**}(x^{**}), y^* \rangle = \langle y - j^{**}(x^{**}), i_z^*(y^*) \rangle$$

and it follows that

$$\|y - i_X^{**}(x^{**})\| = \|y - j^{**}(x^{**})\|.$$

We get $d_Y\left(y, X^{\perp\perp}\right) = d_Z\left(y, X^{\perp\perp}\right)$ and by using the principle of local reflexivity in $Z = \operatorname{span}(X, \{y\})$ we find $d_z\left(y, X^{\perp\perp}\right) = d_z(y, X)$ and thus $d_z(y, X) = d_Y(y, X)$. $\square$

The next key result in the foregoing is as follows:

Theorem 3.9.1. *Let X be a closed subspace of a Banach space Y. If X is an $u-$ ideal in Z for every subspace Z of Y with $\dim Z/X \leq 2$, then X is an u-ideal in Y.*

Proof. We have a possible non-linear $T : Y \to X^{**}$ with $T(x) = x$ for all $x \in X$ such that $\|y - 2i_X{}^{**}T(y)\| = \|y\|$ for all $y \in Y$. For all $y \in Y$, we have that $T(y)$ is a center of symmetry in P_y. Let $y_1, y_2 \in Y$. Let $Z = \operatorname{span}\left(X, \{y_1, y_2\}\right)$. By assumption X is a u-ideal in Z which means that T is linear: $T\left(y_1 + y_2\right) = T\left(y_1\right) + T\left(y_2\right)$ $\square$

3.10 Strict u-Ideals in Banach Spaces

We look at strict u-ideals in Banach spaces, that is ideals for which the Hahn -Banach extension operator is both strict and unconditional. The main aim being an expansion and extension of the research of [23]. A Banach space X is a strict $u-$ideal in its bidual when the canonical decomposition $X^{***} = X^* \otimes Z^{\perp}$ is unconditional. Godfrey, kalton and Saphar [24] observed that the theory of u-ideals is much less satisfactory than in the complex case of h - ideal. Under this we endeavor to fill some gaps in the theory of u-ideals which are strict.

3.10.1 Strict u-Ideals in their Bidual

We use standard Banach space notations. For a Banach space space X, B_X is the closed unit ball and S_X is the unit sphere. The canonical embedding $X \to X^{**}$ is denoted by k_X.

Remark 3.10.1. *l_1 is a $u-$ideal because it is a $u-$ summand in $l_1{}^{**}$ hence it is not a strict $u-$ideal.*

The theorem that follows extends this remark considerably

Theorem 3.10.1. *Suppose that X is a separable Banach space containing l_1. Let P be a contractive projection on X^{***} with $ker P = X^{\perp}$ and such that $V = P\left(X^{***}\right)$ is norming and $\|I - P\| \geq 2$. Then X cannot be a strict u-ideal.*

Proof. Since V is norming the associated operator $T : X^{**} \to X^{**}$ is an isometry. If X contains a copy of l_1, then there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$ and such that $\|x^{**} + x\| = \|x^{**} - x\|$ for all $x \in X$. If $\|I - P\| = a$ so that we can find a net (x_d) in X, converging weak * to Tx^{**}, with

$$\limsup \|Tx^{**} - x_d\| \leq a.$$

Since T is an isometry limsup $\|x^{**} - x_d\| \leq a$ and thus

$$\limsup \|x^{**} + x_d\| = \limsup \|Tx^{**} + x_d\| \leq a.$$

However, $\limsup \|Tx^{**} + x_d\| \geq 2$. $\square$

Proposition 3.10.1. *Let X be either a separable Banach space or a Banach space containing no copy of l_1.*

(1) X is a strict ideal if and only if $\|I - 2\pi\| = 1$, that is if and only if $2 \in G(X)$.

(2) (If X is complex) X is a strict $h-$ ideal in X^{**} *if and only if $\|I - (1 + \lambda)\pi\| = 1$ whenever $|\lambda| \leq 1$ i.e if and only if $G(X) = \{1 + \lambda : |\lambda| \leq 1\}$.*

(3) If X is a strict u-ideal (respectively h-ideal) then every subspace of a quotient space of X is also a strict u-ideal (respectively h-ideal).

Definition 3.10.1. *For every Banach space X we let the natural embedding $k_{X^*} : X^* \to X^{***}$ be an element of $HB\left(X, X^{**}\right)$. And further let $\pi : X^{***} \to X^{***}$ denote the associated ideal projection with* $\ker \pi = X^{\perp}$. *We provide the lemma that follows which is key to the subsequent work.*

Lemma 3.10.1. *Let X be a Banach space containing no copy of l_1. Then $\|I-2\pi\| \leq k_u(X)$.*

Proof. Suppose $x^{**} \in S_{X^{**}}$ then $x^{**} \in Ba(X)$ and so, for $\varepsilon > 0$ arbitrary taken then, there is a series $\sum x_k = x^{**}$ weak * such that for all $-1 \leq \theta_k \leq 1$ and all n

$$\left\| \sum_{k=1}^{n} \theta_k x_k \right\| \leq \liminf_{m\to\infty} \left\| \sum_{k=n+1}^{m} x_k - \sum_{k=1}^{n} x_k \right\| \leq k_u(X) + \varepsilon.$$

Hence if

$$s_n = \sum_{k=1}^{n} x_k \text{ then } \|x^{**} - 2s_n\| \leq k_u(x^{**}) + \varepsilon.$$

It thus follows that $\|I-2\pi\| \leq k_u(X)+\varepsilon$. Since $\varepsilon > 0$, was arbitrary taken $\varepsilon \to 0$ hence we have our desired result. □

The Theorems that follow characterize spaces which are strict u-ideals in their biduals.

Theorem 3.10.2. *Let X be a Banach space containing no copy of l_1. Then X is a strict u-ideal if and only if $k_u(X) = 1$.*

Proof. Suppose X is a strict u-ideal in X^{**}. Then the projection $P = \pi$ and the associated operator $T : X^{**} \to X^{**}$ is the identity. Now if $x^* \in Ba(X)$, then let us select a sequence (x_n) converging weak* to x^{**}. Let further A_n be the convex hull of $\{x_k : k \geq n\}$. If H_n is the weak *-closure of A_n then $\cap_n H_n = \{x^{**}\}$, so that we conclude that

$$k_u(x^{**}) = \|x^{**}\| = 1.$$

Conversely if we suppose that $k_u(X) = 1$ then this direction holds. □

Suppose i_X is the natural embedding $i_X : X \to Y \cdot P_\phi = \phi \circ i_X^*$ is a norm one projection on Y^* with $\ker P = X^\perp$. We say X is an ideal in Y if and only if $HB(X,Y) \neq \emptyset$. When we have $\left\|x^\perp + \phi(x^*)\right\| = \left\|x^\perp - \phi(x^*)\right\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$ we say that X is a $u-$ ideal in Y and ϕ is unconditional. We further note that ϕ is unconditional if and only if $\|I - 2P_\phi\| = 1$. Then we have the well-known notion of $M-$ ideal whose work is extensive that is,

$$\left\|x^{\perp} + \phi\left(x^{*}\right)\right\| = \left\|x^{\perp}\right\| + \left\|\phi\left(x^{*}\right)\right\| \text{ for all } x^{\perp} \in X^{\perp} \text{ and } x^{*} \in X^{*}.$$

Definition 3.10.2. *We say an operator $T_\phi : Y \to X^{**}$ is a norm one operator if*

$$\langle i_X^* y^*, T_\phi(y)\rangle = \langle y, P_\phi\left(y^*\right)\rangle \text{ for all } y \in Y \text{ and } y^* \in Y^*.$$

By this definition it follows that $T_\phi(x) = x$ for all $x \in X$. X is a strict ideal in Y if there is a $\phi \in HB(X, Y)$ such that $\phi\left(X^*\right)$ is norming. In this case ϕ is called strict. Further since $|\langle x^*, T_\phi(y)\rangle| \leq \|T_\phi\| \|x^*\| \|y\|$ we see that ϕ is strict if and only if $T_\phi : Y \to X^{**}$ is isometric.

Proposition 3.10.2. *Suppose X is a strict u-ideal in Y.Then X is a strict $u-$* ideal *in Y if and only if X is a strict u-ideal in span $(X, \{y\})$ for all $y \in Y$.*

Theorem 3.10.3. *X is a strict $u-$ ideal in X^{**} if and only if $\|I - 2\pi\| = 1$.*

Proof. Assume that X is a strict $u-$ ideal in its bidual. Let $x^{**} \in X^{**} \backslash X$. We have $X \cap \left(\cap_{x \in X} B_{X^{**}}\left(x, \|x - x^{**}\|\right)\right) = \phi$ since any element in the intersection would define a norm one projection from span$\left(X, \{x^{**}\}\right)$ onto X which is a contradiction. We thus get $\cap_{x \in X^{**}}\left(x, \|x - x^{**}\|\right) = \{x^{**}\}$ which implies that the only element in $HB\left(X, X^{**}\right)$ is k_{X^*} Since X^* is norming for X^{**} the other direction is trivial. □

The Theorem that follows was first inspired by Theorem 5.5 in Godefry [24]. Vegard lima and Asvald Lima [42] in Theorem 2.9 removed the assumption that the space does not contain l_1 as an improvement. We further show that strict u-ideals are separably determined.

Proposition 3.10.3. *Let X be a Banach space which is a strict u-ideal in its bidual. If Y is any separable subspace of X then it is a strict u - ideal in its bidual and is X separably determined.*

Proof. Let Y be a closed subspace of X with natural embedding $i_X : Y \to X$. Assume the ideal property $\|I - 2\pi_X\| = 1$ where $\pi_X = k_X K_X{}^*$. We need to show that $\|I - 2\pi_Y\| = 1$ where $\pi_Y = k_{Y^*} K_Y{}^*$. Indeed $i_Y{}^{**} k_Y = k_X i_Y$ and $i_Y{}^{***} k_{X^*} = k_Y * i_Y{}^*$ so that $i_Y{}^{***} \pi_X = \pi_Y i_Y{}^{***}$. Now we get

$$1 \geq \left\| i_Y{}^{***} \left(I - 2\pi_X \right) \right\| = \left\| \left(I - 2\pi_Y \right) i_Y{}^{***} \right\| .$$

Since $i_Y{}^{**} : Y^{**} \to X^{**}$ is isometric, it is onto Y^{**} hence $\|I - 2\pi_Y\| = 1$. Therefore, strict u-ideals are separably determined. $\square$

CHAPTER FOUR

SPACES OF IDEAL OPERATORS

4.1 Introduction

In this chapter, we determine some important spaces of ideal operators and ideal characteristics. Special consideration is given to Frechet spaces, Spaces of finite rank operators and spaces of Hahn-Banach extension operators. The characteristics of ideals and related properties in these spaces as well as in some of their dual spaces are obtained.

4.2 Frechet Space of Operator Ideals

The notions in both Hilbert and Banach spaces can be generalized if a Frechet space say F is taken an ambient space. Therefore, this section considers the ideal properties in the Frechet spaces with respect to the approximate identities, density and smoothness. Bounded approximate identity is a key concept in the theory of amenability of algebras. We show that algebra of compact operators on Frechet space X has both the right and left locally bounded approximate identities. Sufficient conditions for the existence of these identities are established based on the geometry properties of the Frechet space X and its dual space X^* respectively.

A topological linear space X is referred to as a *lcs* if it has a local neighbourhood base comprising convex sets. The *lcs* X is referred to as reflexive if it coincides with the continuous dual of its continuous dual space, that is $X = X^*$. A *lcs* is called a metrizable *lcs* if it possesses countable local neighbourhood base. A Frechet space X is a complete, metrizable *lcs*. Its notions therefore generalize Banach space and Hilbert spaces. Any algebra $\mathbb{A}$ equipped with a structure of *lcs* with respect to which the product is separately continuous is a topological algebra[8]. So, a Frechet algebra is a complete topological algebra of which an increasing countable collection $\{p_i;\ i \in \mathbb{N}\}$ of sub-multiplicative continuous semi-norms determines its topology. A Frechet algebra $\mathbb{A}$ is called amenable if given an $\mathbb{A}-$bimodule Y , every continuous derivation from $\mathbb{A}$ to the dual bimodule Y^* is inner.

Given *lcs* X and Y . Let $T : X \to Y$ be a linear operator. Then, $T : X \to Y$ is called bounded if for some neighbourhood U in X, $T(U)$ is bounded in Y and the operator ideal $U(X;Y)$ is closed if $U = \bar{U}$.

A space X is said to have an unconditional partition of the identity (UPI) if for a sequence $\{T_n\}_n$ of continuous linear operators $T_n : X \to X$ we have $dimT_n(X)$ finite and $\sum_i T_i s$, where convergence is unconditional, $s \in X$.

The next results then follow:

Proposition 4.2.1. *Suppose X^* is the dual of a Frechet space X, there exists a bijection between operators on X^* and the strict inductive limit of the inductive system of continuous linear operators of Banach spaces.*

Proof. Let $i, j \in I$ such that $j \geq i$, we define a map $f_{ij} : X_i \longrightarrow X_j$ such that $U_i \subset U_j$ where U_i and U_j are 0-neighbourhoods in X_i and X_j respectively and f_{ij} is continuous with $\{X_i\}_i$ being family of Banach spaces. Hence, we identify X^* as the strict direct limit of sequence of Banach spaces $\{X_i\}$. That is $X^* = \lim \longrightarrow X_i = \bigcup X_i = X^* \quad (i = 1, 2, \ldots)$ with $f_{ij} \circ f_{jk} = f_{ik}$ satisfied for $j \geq i, k \geq j. X^*$ is endowed with strict inductive limit topology where $f_i : X_i \longrightarrow X^*$ is continuous such that $f_i(s_i) = s^*$ and $f_{ij}(s_i) = s_j$. Hence, X^* is a complete lcs. We identify X^* as the dual of a Frechet space X. Moreover, given $i \in I$ and $T_i : D(T_i) \subset X_i \longrightarrow X_i$, then $\{T_i : i \in I\}$ can be seen as an inductive system of operators in such a way that for $s_i \in D(T_i) \subset X_i$ and $i > j$,

$$T_i(f_{ji}(s_j)) = f_{ji}(T_j(s_j)).$$

We then define T^* as the inductive limit of the inductive system $\{D(T_i) : i \in I\}$ using $T^*(s^*) = f_i(T_i(s_i))$ or $f_i^{-1}(T^*(s^*)) = T_i\left(f_i^{-1}(s^*)\right)$ where $s^* \in D(X^*)$ with $i \in I$.

Therefore, we refer to T^* as the direct limit of $\{T_i : i \in I\}$. We have that T^* is a linear operator. Hence, for each $i, T_i \in L(X_i)$, there exists $T^* \in L_I(X^*)$. In the sequel, we finally have the following relation. $T \in L_I(\lim_{\longleftarrow} X_i) = L_I(X)$ and T^* $L_I(\lim_{\longrightarrow} X_i) = L_I(X^*)$.

$\square$

Proposition 4.2.2. *Suppose X and Y are Frechet spaces where X_0 and Y_0 are subspaces of X and Y respectively. Let X be quasi normable and Y be reflexive. If $R \in L_I(X_0, X) \subseteq L(X_0, X)$ and $S \in M_I(Y, Y_0) \subseteq L(Y, Y_0)$, then the algebra of compact operators $K_I(X, Y)$ is an ideal in $L_I(X, Y)$.*

Proof. Suppose $R \in L_I(X_0, X), S \in M_I(Y, Y_0)$ and $T \in K_I(X, Y)$. We need to show that $K_I(X, Y)$ is an ideal. Since $K_I(X, Y) \subseteq L_I(X, Y)$, it is not empty. By definition, there exists some neighbourhood $U_0 \subset X_0$ and a bounded subset $B \subset X$ such that

$$RU_0 \subset B \tag{4.1}$$

Since X is quasi normable, there are 0-neighbourhoods U and V with $V \subset U$ such that for every $\epsilon > 0$ we have $V \subset B + \epsilon U$. Hence, by definition there exists a compact set $W \subset Y$ where TV is relatively compact in Y. That is

$$TV \subset W. \tag{4.2}$$

Lastly, since Y is reflexive, the relatively compact set TV is a bounded set in Y. Hence, there exists by definition a compact set $G \subset Y_0$ where $S(TV)$ is relatively compact in Y_0. That is

$$S(TV) \subset G. \tag{4.3}$$

From relations (4.1) and (4.2), $RU_0 \subset B + \epsilon U$. Hence,

$$T(RU_0) \subset W. \tag{4.4}$$

From relations (4.3) and (4.4), since $T\left(RU_0\right)$ is relatively compact, which implies that it is bounded in a reflexive Frechet space Y. Hence,

$$ST\left(RU_0\right) \subset G.$$

This implies that $STR \in K_I\left(X_0, Y_0\right)$. Therefore, $K_I(X, Y)$ is an ideal.

$\square$

Proposition 4.2.3. *Suppose X is a Frechet space. An UPI for X implies an UPI for X^*.*

Proof. Let a Frechet space X have UPI. That is, for a continuous linear sequence of operators $\{T\}_i \subset L_I(X)$ with $\dim\left(T_i(s)\right) < \infty$ and $i \in \mathbb{N}$, we have $\sum_i^n T_i(s) = s$. Let s_i converge to s weakly in X. Then, we have

$$\sum_i T_i\left(s_i - s\right) = \sum_i \left(T_i s_i - T_i s\right) \longrightarrow 0.$$

Therefore, for all k there exists j and $c > 0$ such that

$$\sum_i p_k\left(T_i\left(s_i - s\right)\right) \leq c p_j\left(s_i - s\right).$$

Therefore,

$$\begin{aligned} \left|c p_j\left(s_i\right) - c p_j(s)\right| &\geq \left|\sum_i p_k\left(T_i\left(s_i\right)\right) - \sum_i p_k\left(T_i(s)\right)\right| \\ &\leq \sum_i p_k\left(T_i\left(s_i - s\right)\right) \leq c p_j\left(s_i - s\right), \end{aligned}$$

Hence,

$$\left|\sum_i p_k\left(T_i\left(s_i\right)\right) - \sum_i p_k\left(T_i(s)\right)\right| \leq c p_j\left(s_i - s\right).$$

Then for $\lambda \in X^*, s \in X$ we have

$$|cp_k^*(\lambda)p_j(s_i)| - |cp_k^*(\lambda)p_j(s)| \geq \sum_i |\lambda \left(T_i(s_i)| - \sum_i |\lambda \left(T_i(s) \right| \leq \sum_i \right| \lambda \left(T_i \left(s_i - s \right) \right)$$
$$\leq cp_k^*(\lambda)p_j(s_i - s).$$

That is

$$\sum_i |\lambda(T_i(s_i))| - \left| \lambda \sum_i T_i(s) \right| \leq cp_k^*(\lambda)p_j(s_i - s).$$

Since the summation is over i and also since X has UPI, i.e. $\sum_i T_i s = s$, we then have

$$\sum_i | \lambda(T_i(s_i)) | - |\lambda \sum_i T_i(s) | \leq cp_k'(\lambda)p_j(s_i - s)$$

then,

$$\sum_i |\lambda(T_i(s_i))| - |\lambda(s)| \leq cp_k^*(\lambda)p_j(s_i - s) \tag{4.5}$$

Let define an operator $T_i^* : X^* \to X^*$ such that $T_i'(\lambda s_i) = \lambda(T_i(s_i))$ where $\lambda \in X^*$. (4.5) now becomes

$$\sum_i |T_i^*(\lambda s_i)| - |\lambda(s)| \leq cp_k^*(\lambda)p_j(s_i - s)$$
$$\implies |cp_k^*(\lambda)p_j(s_i)| - |cp_k^*(\lambda)p_j(s)| \geq \sum_i |T_i^*(\lambda s_i)| - |\lambda(s)|$$
$$\leq \left| \sum_i T_i^*(\lambda s_i) - \lambda(s) \right| \leq cp_k^*(\lambda)p_j(s_i - s).$$

This implies that

$$\left| \sum_i T_i^*(\lambda s_i) - \lambda(s) \right| \leq cp_k^*(\lambda)p_j(s_i).$$

Let $\epsilon = \max(m, cp_k^*(\lambda)p_j(s_i))$ for $m > 0$

$$\therefore \left| \sum_i T_i^*(\lambda s_i) - \lambda(s) \right| \leq \epsilon.$$

Hence, it implies that

$$\sum_i T_i^* (\lambda s_i) - \lambda s \longrightarrow 0.$$

For $s \in X$, we have

$$\sum_i T_i^*(\lambda s) - \lambda s = 0.$$

that is,

$$\sum_i T_i^*(\lambda s) = \lambda s \text{ for } \lambda(s) \in X^*.$$

Therefore X^* has an UPI.

□

Next, we restrict our attention to the separable Banach space setting and investigate certain approximations.

Let X be a separable Banach space. We say that a sequence of compact operators $K_n : X \to X$ is a compact approximating sequence if $\lim_{n\to\infty} K_n x = x$ for every $x \in X$. So $(K_n), n \in \mathbb{N}$ is an approximating sequence if each K_n is finite-rank operator. We extend the notion in [40], we say that a Banach space X has (UKAP) if there is a compact approximating sequence $K_n : X \to X$ such that $\lim_{n\to\infty} \|I - 2K_n\| = 1$. For the complex case of a complex Banach space we say that X has a complex (UKAP) if there is a compact approximating sequence such that $\lim_{n\to\infty} \|I - (1+\lambda)K_n\| = 1$ whenever $|\lambda| = 1$. This condition imposes the ideal property.

Lemma 4.2.1. *[24]*

(i) Let X be a separable Banach space.Then X has $(UKAP)$ if and only if for every $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that for every $x \in X$ and every n and every $\theta_j = \pm 1, 1 \le j \le n$, we have $\sum_{n=1}^{\infty} A_n x = x$ and $\left\| \sum_{j=1}^{n} \theta_j A_j x \right\| \le (1+\varepsilon)\|x\|$.

(ii) Let X be a separable complex Banach space. Then X has complex (UKAP) if and only if for every $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that for

every $x \in X$ and every n and every $|\theta_j| \leq 1$, for $1 \leq j \leq n$, we have $\sum_{n=1}^{\infty} A_n x = x$ and $\left\| \sum_{j=1}^{n} \theta_j A_j x \right\| \leq (1+\varepsilon)\|x\|$.

Proposition 4.2.4. *Let X be a separable Banach space. If X has (UKAP) then X is a $u-$ ideal and $K(X)$ is a $u-$ ideal in $L(X)$.*

Proof. The fact that X is a $u-$ ideal follows easily from the previous chapter. The remainder conclusion also follows since if z is a finite-dimensional subspace of $K(X)$ and $\varepsilon > 0$.

We can find K compact such that $\|KS - S\| \leq \epsilon \|S\|$ for $S \in Z$ and $\|1 - 2K\| \leq 1 + \epsilon$. Then consider $\Lambda(S) = KS$ for $S \in L(X)$. $\square$

In the Theorems that follow we show the converse of the above proposition under the condition of separability and reflexivity.

Theorem 4.2.1. *Let X be separable reflexive Banach space. Then X has UKAP if and only if $K(X)$ is a $u-$ ideal in $L(X)$.*

Proof. We denote by $P : L(X)^* \to L(X)^*$ the projection with $kerP = L(X)^{\perp}$ and by $T : L(X)^{\cdot} \to K(X)^{**}$ the induced operator. For $x \in X$ and $x^* \in X^*$, we let $x \otimes x^* \in K(X)^*$ be the linear functional given by $\langle K, x \otimes x^* \rangle = \langle Kx, x^* \rangle$. This functional has a natural extension to $L(X)$, also denoted by $x \otimes x^*$. Let $u \in X$ and $v \in X^*$ be points of Frechet smoothness with $\|u\| = \|v^*\| = 1$. We suppose $u^* \in S_{X^*}$ and $v^{\cdots} \in S_X$ satisfy $u^*(u) = v^*(v) = 1$. Let $A : X \to X$ be the rank-one operator given by $Ax = v^*(x)u$. Then A is a point of Frechet smoothness in $L(X)$. For real λ we have $\|A + \lambda I\| = 1 + \lambda u^*(v) + \alpha(|\lambda|)$. Hence in $K(X)^{**}$ we obtain $\|A + \lambda T(I)\| \leq 1 + \lambda u^*(v) + \alpha(|\lambda|)$ in particular, $\langle v \otimes u^*, A + \lambda T(I) \rangle \leq 1 + \lambda u^*(v) + \alpha(|\lambda|)$. Hence $\langle v \otimes u^*, T(I) \rangle = u^*(v)$. Now since the points of Frechet smoothness form a dense G_δ in both X and X^* we have for every $\in X, x^* \in X^*, \langle x \otimes x^*, T(I) \rangle = x^*(x)$.

Now by Lemma 2.2 in [23], there is a net (K_d) in $K(X)$ such that K_d converges *weak** to $T(I)$ and limsup $\|I - 2K_d\| = 1$. But then $K_d \to I$ for the weak operator topology.

Hence for each d we can find $L_d \in \operatorname{co}\{K_e : e \geq d\}$ such that $K_d \to I$ for the strong operator topology. It follows that there is a compact approximating sequence (M_n) in $K(X)$ such that $\lim \|I - 2M_n\| = 1$. $\square$

Theorem 4.2.2. *Let X be a complex Banach space such that X^* is separable. Then X has complex (UKAP) if and only if $K(X)$ is an h-ideal in $L(X)$ and X is an h-ideal.*

Proof. For $x^* \in X^*$ and $x^{**} \in X^{**}$ we use $x^* \otimes x^{**}$ to denote the element of $K(X)^*$ given by $\langle S, x \otimes x^* \rangle = x^{**}(S^* x^*)$. It is easy to note that the formula then defines $x^* \otimes x^{**} \in L(X)^*$. We define $L(X) \to K(X)^{**}$. By this it is possible to define an operator $H : X^{**} \to X^{**}$ such that $\langle x \otimes x^*, T(I) \rangle = \langle x^*, Hx^{**} \rangle$, we first argue that H is Hermitian. In fact suppose $|\lambda| = 1$. It follows from the fact that P is Hermitian that $\|1 - (1+\lambda)H\| \leq 1$. Thus if ϕ is a state on $L(X)^{**}$ then $|1 - (1+\lambda)\phi(H)| \leq 1$. Hence $\phi(H)$ is real and further $0 \leq \phi(H) \leq 1$. Hence H is a Hermitian.
Next we argue that there is an Hermitian $H_0 : X \to X$ such that $H = {H_0}^{**}$. In fact for any real $t, e^{(itH)}$ is an isometric isomorphism on X^{**}. We recall that X is necessarily a strict $h-$ ideal by Theorem 6.6 in [24]. Applying Theorem 5.7 in [24] we can deduce that $e^{(itH)}$ maps X to X and it is weak * continuous. On differentiating we conclude that $H = {H_0}^{**}$ where $H_0 : X \to X$ is the restriction of H.

Next we recall that the collection of points of Frechet smoothness in X form a dense G_δ as do the points of Gateaux smoothness in X^*. Let us suppose that $u \in S_X$ is a point of Frechet smoothness and that $u^* \in S_{X^*}$ is a point of Gateaux smoothness and v^{**} is the corresponding exposed functional in $S_{X^{**}}$. We define the rank-one operator $Ax = v^*(x)u$ and claim that

$$\|A + \xi I\| = 1 + \mathcal{R}\xi\left(v^{**}\left(u^*\right)\right) + O(|\xi|).$$

In fact

$$\|A + \xi I\| \geq \mathcal{R}v^{**}\left(A^* u^* + \xi u^*\right) = 1 + \mathcal{R}\left(\xi v^{**}\left(u^*\right)\right).$$

Conversely, for any ξ we may pick

$$x^*(\xi) \in S_{X^*}\text{so that} 0 \leq x^*(\xi)(u) \leq 1 \text{and} \left\|\left(A^* + \xi I\right)\left(x^*(\xi)\right)\right\| \geq \|A + \xi I\| - |\xi|^2.$$

Letting $\xi \to 0$ we observe that if x^* is any weak* cluster point then $(0 \leq x^*(u)) \leq 1$ and $\|Ax^*\| = 1$. Hence $\lim \xi \to 0 \text{as} x^*(\xi) = u^*$ weak*. However, this implies, since u is a point of Frechet smoothness, that $\lim \xi \to 0 \, \|x^*(\xi) - u^*\| = 0$. It now follows immediately from the Gateaux smoothness of the norm at v^* that

$$\|A + \xi I\| = 1 + \mathcal{R}\left(\xi v^{**}\left(u^*\right)\right) + O(|\xi|).$$

Using the formal identity

$$\|T(A) + \xi T(I)\| \leq 1 + \mathcal{R}\left(\xi v^{**}\left(u^*\right)\right) + O(|\xi|)$$

and hence

$$\mathcal{R}\left\langle\left(A^{**} + \xi H\right) v^{**}, u^*\right\rangle \leq 1 + \mathcal{R}\left\langle\xi v^{**}, u^*\right\rangle + O(|\xi|).$$

It follows that $\langle Hv^{**}, u^*\rangle = \langle v^{**}, u^*\rangle$. If we fix u^*, the collection of all v^{**} as v^* ranges all over all points of Gateaux smoothness spans a weak *-dense subspace of X^* and so $H_0 = I$. Now there is a net (K_d) in $K(X)$ such that K_d converges weak * to $T(I)$ and limsup $\|I - (1 + \lambda)K_d\| = 1$ whenever $|\lambda| = 1$. □

4.3 Locally Uniform Rotundity

The properties of locally uniform rotund norms are useful in showing that sufficiently many simple tensors for example $x^* \otimes y^{**}$ when viewed as functionals on the finite rank operator $F(X, Y)$ have unique Hahn-Banach extensions to the space of all bounded operators $K(X, Y)$.

Some of the basic facts on locally uniform rotund norms useful in the sequel are:

Definition 4.3.1. *The norm on a Banach space Y is locally uniformly rotund at a point $y \neq 0$ if $\lim_n \|y - y_n\| = 0$ whenever $(y_n) \subseteq Y$ with $\|y_n\| = \|y\|$ for all n and $\lim_n \left\|\frac{y+y_n}{2}\right\| = \|y\|$. The norm is locally uniformly rotund if it is locally uniformly rotund at every point $y \neq 0$ in Y.*

The result that follows provides the link between the denting points and the norm rotund characteristic:

Proposition 4.3.1. *Let Y be a Banach space and let $y \in Y \backslash \{0\}$. The following are equivalent:*

(a) The norm $\|.\|$ is locally uniformly rotund at y.

(b) If $(y_n) \subseteq Y$ is such that $\lim_n \left(2\|y\|^2 + 2\left\|y_n\right\|^2 - \left\|y + y_n\right\|^2 = 0\right)$,
then $\lim_n \|y - y_n\| = 0$.

(c) The function $\delta_y : [0, 2\|y\|] \to [0, \|y\|]$,defined by the formula $\delta_y(\varepsilon) = \inf\{\|y\| + \|u\| - \|y + u\| : \|u\| = \|y\|, \|y - u\| \geq \varepsilon\}$,satisfies $\delta_y(\varepsilon) > 0$ whenever $0 < \varepsilon \leq 2\|y\|$.

We provide the following renorming result whose proof can be given using the above proposition.

Lemma 4.3.1. *Let Y be a Banach space with $\|.\|$ and let $y \in Y \backslash \{0\}$. Assume $\|. \mid \|$ is an equivalent norm on Y such that $\|.\| \mid$ is locally uniformly rotund at y. Let $a, b > 0$ and define a new norm $|.|$ on Y by $|y|^2 = a\|y\|^2 + \||y|\|^2$. Then $|.|$ is locally uniformly rotund at y.*

Remark 4.3.1. *From the lemma above if we choose a close to 1 and b close to 0 , we may assume that the new norm is "close" to the original norm. Precisely this leads to the next lemma.*

Lemma 4.3.2. *Let Y be a Banach space, let $Z \subseteq$ Y be a closed separable subspace and let $\varepsilon > 0$ be given. There exists an equivalent norm $|||.|||$ on Y such that $B_Y(0,1) \subseteq B_{(Y,|||.\,|\,||)}(0,1) \subseteq B_Y(0,1+\varepsilon)$ and such that the norm $\|..\|$ is locally uniformly rotund at every point $z \neq 0$ in Z.*

Proof. Z is a separable space and so by theorem 11.2.6 in [46]. It has an equivalent locally uniform rotund norm which can be extended to an equivalent norm on Y in such a way that this new norm is locally uniformly rotund at every $z \neq 0$ in Z.

Let $\|y\|$ be this equivalent norm on Y which is locally uniformly rotund at every point $z \in Z$, $z \neq 0$. For some $c \geq 1$, we have $\frac{1}{c}\|y\| \leq |||y||| \leq c\|y\|$ for all $y \in Y$. If $1/c^2 > \theta > 0$ and we let $|y|_\theta^2 = (1-\theta c^2)\,\|y\|^2 + \theta|||y|||^2$, then $|\cdot|_\theta$ is an equivalent norm on Y. $|\cdot|_\theta$ is locally uniformly rotund at every $z \neq 0$ in Z. Let $\varepsilon > 0$ be given, choose θ_0 so small that $\sqrt{1-\theta_0 c^2 + \theta_0/_{c^2}} \geq (1+\varepsilon)^{-1}$. Finally, we redefine $\|.\| = \|.\|_\theta$. and get $(B_{(Y,\|\cdot\|)}(0,1) \subseteq B_{(Y}, |||\cdot|||)(0,1) \subseteq B_{(Y,\|\cdot\|)}(0,1+\varepsilon)$ as desired.

$\square$

A similar result as the one above for subspaces of dual spaces will be of interest to us too. We shall consider finite-dimensional subspaces. This is considered in the Lemma that follows.

Lemma 4.3.3. *Let Y be a Banach space, let $F \subseteq Y^*$ be finite-dimensional subspace and let $\varepsilon > 0$. There exists an equivalent norm $\|.\|$ on Y such that*

$$(B_Y(0,1) \subseteq B_{(Y}, |||.|)(0,1) \subseteq B_Y(0,1+\varepsilon)$$

and such that the dual norm of $|||.|||$ on Y^* is locally uniformly rotund at every point $y^* \neq 0$ in F

Proof. Let $\theta > 0$ be given. Since $\dim F < \infty$, there exists an equivalent locally uniformly rotund norm $|.|$ on F. Moreover we may assume

$$B_F(0,1) \subseteq B_{(F,|,|)}(0,1) \subseteq B_F(0,1+\theta)\cdot \text{Let } B = \text{Conv}\left(B_{Y^*}, B_{(F,|\cdot|)}\right)$$

and let $|.|$ be the norm on Y^* defined by B. B is weak *-compact, so $|.|$ is a dual norm. Moreover,

$$|y^*| \leq \|y^*\| \leq (1+\theta)\,|y^*| \ \text{ for all } y^* \in Y^* \text{ or } B_{Y^*}(0,1) \subseteq B_{(Y^*,|\cdot|)}(0,1) \subseteq B_{Y^*}(0,1+\theta)\cdot$$

Next we define a new norm on Y^* by $|||y^*|||_\theta^2 = |y^*|^2 + \theta d^2\,(y^*,F) \cdot d\,(y^*,F)$ is computed in the $|.|$ - norm on $Y^* \cdot (Y^*, |||.|||_\theta)$ is locally uniformly rotund at every point $y^* \neq 0$ in F. We shall show that $\|\,|.||_\theta$ is a dual norm. Assume $y^*_\alpha \to y^*$ weak* with $|||y^*_\alpha|||_\theta \leq 1$. Choose $(f^*_\alpha) \subseteq F$ such that $|y^*_\alpha - f^*_\alpha| = d\,(y^*_\alpha, F)$ (f^*_α) is a bounded net and $\dim F < \infty$, so by passing to a subnet, we may assume ${f^*}_\alpha \to f^* \in F$ in norm. We get

$$d\,(y^*_{\alpha^*}, F) \leq |y^* - f^*| \leq \liminf_\alpha \mid y^*_\alpha - f^*_\alpha \mid = \liminf_\alpha d\,(y^*_\alpha, F)\,.$$

Hence,

$$|||y^*|||_\theta^2 = |y^*|^2 + \theta d^2\,(y^*,F) \leq \liminf_\alpha |y^*_\alpha|^2 + \theta \liminf_\alpha d^2\,(y^*_\alpha,F) \leq \liminf_\alpha |||y^*_\alpha|||_\theta^2\,.$$

This shows that $\|.\mid\|_\theta$ is a dual norm. Now since
$d\,(y^*_\alpha, F) \leq |y^*|$, we get $|y^*| \leq \,||y^*|||_\theta \leq (1+\theta)^{1/2}\,|y^*|$.
Thus

$$(1+\theta)^{-1}\,\|y^*\| \leq |||y^*|||_\theta \leq (1+\theta)^{1/2}\,\|y^*\|\,.$$

This implies that

$$(1+\theta)^{-1/2}\|y\| \leq |||y^*|||_\theta \leq (1+\theta)^{-1}\|y\| \text{ for all } y \in Y.$$

Let $\varepsilon > 0$. If we use $|||.||| = (1+\theta)^{-1}|||.|||_\theta$ as the new norm, and choose θ such that $(1+\theta)^{3/2} \leq 1+\varepsilon$, then $(1+\theta)^{-3/2}\|y\| \leq |||y|||_{-1} \leq \|y\|$ for all $y \in Y$ and thus

$$B_Y(0,1) \subseteq B_{(Y,\,|||\cdot|||\big)}\,(0,1) \subseteq B_Y(0,1+\varepsilon).$$

$\square$

Lemma 4.3.4. *[43] Let X and Y be Banach spaces.Let $x^* \otimes y \in \mathcal{F}(Y,X)^*$ with $x^* \in X^*$ and $y \in Y$.If the norm of Y is locally uniformly rotund at y, then the only Hahn-Banach extension of $x^* \otimes y$ to $\mathcal{H}(Y,X)$ is the trivial one, that is $HB\,(x^* \otimes y) = \{x^* \otimes y\}$.*

We shall also need elements in the bidual and the lemma that follows takes care of that.

Lemma 4.3.5. *[47] Let X and Y be Banach spaces. Let $y^* \otimes x^{**} \in \mathcal{F}(X,Y)^*$ with $y^* \in Y^*$ and $x^{**} \in X^{**}$. If the norm of Y^* is locally uniformly rotund at y^* then the only Hahn-Banach extension of $y^* \otimes x^{**}$ to $\mathcal{H}(X,Y)$ is the trivial one, that is $HB\,(y^* \otimes x^{**}) = \{y^* \otimes x^{**}\}$.*

4.4 Renorming and the Hahn-Banach Extension Operators

Let X and Y be Banach spaces. We let $\mathcal{A}$ and $\mathcal{B}$ denote closed operator ideals. This will ensure that $\mathcal{F}(Y,X) \subseteq \mathcal{A}(Y,X)$ and that $\mathcal{A}(Y,X)$ is a closed subspace of $\mathcal{H}(Y,X)$.

Theorem 4.4.1. *Let X be a Banach space and Z be a separable subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(\widehat{Y},X)$ is an ideal in $\mathcal{B}(\widehat{Y},X)$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(Y,X),\mathcal{B}(Y,X))$ such that $(\psi\,(x^* \otimes z))\,(T) = (x^* \otimes z)\,(T)$ for all $x^* \in X^*, z \in Z$, and $T \in \mathcal{B}(Y,X)$.*

Proof. For every $\varepsilon > 0$ we can find an equivalent norm $\|\varepsilon\|$ on Y such that $Y_s = (Y, \|.\|_s)$ is locally uniformly rotund at every $z \neq 0$ in Z_ε and such that $B_Y(0,1) \subseteq B_{Y_\varepsilon}(0,1) \subseteq B_Y(0,1+\varepsilon)$. By assumption, $\mathcal{A}\,(Y_\varepsilon, X)$ is an ideal in $\mathcal{B}\,(Y_\varepsilon, X)$ so there exists a Hahn-Banach extension $Q_\varepsilon : \mathcal{A}\,(Y_\varepsilon, X)^* \to \mathcal{B}\,(Y_\varepsilon, X)^*$. Now, $(Q_\varepsilon\,(x^* \otimes z))\,(T) = (x^* \otimes z)\,(T)$ for all $x^* \in X^*$, all $z \in Z$, and all $T \in \mathcal{B}\,(Y_\varepsilon, X)$. Let $I_s : Y_s \to Y$ denote the identity mapping. Then $\|I_\varepsilon^{-1}\| = 1$ and $\|I_\varepsilon\| \to 1$ as $\varepsilon \to 0$. Define $\psi_s \in \mathcal{H}\,(\mathcal{A}(Y,X)^*, \mathcal{B}(Y,X)^*)$ by

$$(\psi_s(\phi))\,(T) = (Q_s)\,(\phi_s)\,(T \circ I_s)\,, \phi \in \mathcal{A}(Y,X)^*, T \in \mathcal{B}(Y,X)$$

where $\phi_s \in \mathcal{A}(Y_s, X)^*$ is defined by

$$\phi_\varepsilon(S) = \phi\left(S \circ I_\varepsilon^{-1}\right), \quad S \in \mathcal{A}(Y, X).$$

We can conclude that $(\psi_\varepsilon), \varepsilon \in (0,1]$ has a subnet converging weak* to some $\psi \in HB(\mathcal{A}(Y,X), \mathcal{B}(Y,X))$ with required property. $\square$

The following lemma and Theorem are crucial in the proof of one of our main results

Lemma 4.4.1. *[47] Let X be a Banach space and Z be a separable subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. The subset*

$$K_Z = \psi \in H\ B(\mathcal{A}(Y,X),\ \mathcal{B}(Y,X)) : (\psi(x^* \otimes z))(T) = (x^* \otimes z)(T),\ \forall\ x^*\ \in X^*, z \in Z$$

and $T \in \mathcal{B}(Y,X)\}$ of $\left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$ is compact in the weak *-topology.

Theorem 4.4.2. *Let X and Y be Banach spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(\widehat{Y}, X)$ is an ideal in $\mathcal{B}(\widehat{Y}, X)$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(Y,X), \mathcal{B}(Y,X))$ such that $(\psi(x^* \otimes y))(T) = (x^* \otimes y)(T)$ for all $x^* \in X^*, y \in Y$, and $T \in \mathcal{B}(Y,X)$.*

Proof. See [43] $\square$

We now state some of our main results:

Proposition 4.4.1. *If $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{H}(Y,X)$ for every separable Banach space Y, then X has a metric approximation property.*

Proof. Let $L \subseteq X$ be a separable subspace, then we can find a separable ideal Y in X with $L \subseteq Y$ and $\varphi : Y^* \to X^*$ be a Hahn-Banach extension operator. By having $\psi : \mathcal{F}(Y,X)^* \to \mathcal{H}(Y,X)^*$ be a Hahn-Banach extension operator with $\psi(x^* \otimes y) = x^* \otimes y$ for all $y \in Y$ and $x^* \in X^*$ and $I : Y \to X$ be the identity map. We see that $\psi^*(I) \in \mathcal{F}(Y,X)^{**}$ there exists a net $(T_\alpha) \subseteq \mathcal{F}(Y,X)$ such that $\sup_\alpha \|T_\alpha\| \leq \|I\| = 1$ and $T_\alpha \to \psi^*(I)$ in the weak *-topology.

In particular

$$\langle T_\alpha y, x^*\rangle = \langle T_\alpha, x^* \otimes y\rangle \to \langle \psi^*(I), x^* \otimes y\rangle = \langle I, \psi\left(x^* \otimes y\right)\rangle = \langle Iy, x^*\rangle \text{ for all} y \in Y$$

and $x^* \in X^*$, that is $T_\alpha \to I$ in the weak operator topology. By taking a new net from $\mathrm{Conv}\,(T_\alpha)$, which we also denote by (T_α), we may assume that $T_\alpha \to I$ in the strong operator topology.

Let $\widehat{T}_\alpha = T_\alpha{}^{***\circ}\varphi^*|_X \in \mathcal{F}(X, X)$, then $\left\|\widehat{T}_\alpha\right\| = \|T_\alpha\| \leq 1$ and $\widehat{T}_\alpha$ converges pointwise to the identity I_X on Y. It follows then that X has the metric approximation property. □

Proposition 4.4.2. *If $\mathcal{F}(\hat{X}, X)$ is an ideal in $\mathcal{H}(\hat{X}, X)$ for every equivalent renorming $\hat{X}$ of , then X has a metric approximation property.*

Proof. Let $Y = X$. Then there exists $\psi \in HB(\mathcal{F}(X, X), \mathcal{H}(X, X))$ such that $\psi\left(x^* \otimes x\right) = x^* \otimes x$ for all $x \in X$ and $x^* \in X^*$. Let $i : \mathcal{F}(X, X) \to \mathcal{H}(X, X)$ be the natural inclusion and define $P = \psi \circ i^*$. Using Theorem 5.4 from [24] with P as the ideal projection we conclude that X has the metric approximation property. Let I_X be the identity operator on X, it can be seen that $\psi^*\left(I_X\right) \in \mathcal{F}(X, X)^{**}$. It follows that there exists a net $(T_\alpha) \subseteq \mathcal{F}(X, X)$ such that $\sup_\alpha \|T_\alpha\| \leq= 1$ and $T_a \to I_X$ in the weak *-topology. In particular $\langle T_\alpha x, x^*\rangle \to \langle \psi^*\left(I_X\right), x^* \otimes x\rangle = \langle I_X, \psi\left(x^* \otimes x\right)\rangle = \langle I_X x, x^*\rangle$ for all $x \in X$ and $x^* \in X^*$, that is $T_\alpha \to I_X$ in the weak operator topology. By taking a new net consisting of convex combinations of the T_α 's we may assume that the net converges in the strong operator topology. □

4.5 Dual Renorming and the Hahn-Banach Extension Operators

In this section we proof our major result under dual renorming. We shall replace a separable subspace $Z \subseteq Y$ with a finite-dimensional subspace $F \subseteq Y^*$.

Proposition 4.5.1. *Let X and Y be Banach spaces, and let F be a finite-dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X, \hat{Y})$ is an ideal* *in* $\mathcal{B}(X,\hat{Y})$ *for every equivalent renorming* $\hat{Y}$ *of* Y*, then there exists a* $\psi \in HB(\mathcal{A}(X,Y),\mathcal{B}(X,Y))$ *such that* $\left(\psi\left(y^{*}\otimes x^{**}\right)\right)(T)=\left(y^{*}\otimes x^{**}\right)(T)$ *for all* $y^{*}\in F$*, all* $x^{**}\in X^{**}$ *and all* $T\in\mathcal{B}(X,Y)$*.*

Proof. For all $\varepsilon\in(0,1]$ there exists by an equivalent norm $\|.\|_{\varepsilon}$ on Y such that the dual norm on Y^{*} is locally uniformly rotund at every point $y^{*}\neq 0$ in Y such that

$$B_{Y}(0,1)\subseteq B_{(Y,\|\|\|_{\varepsilon})}(0,1)\subseteq B_{Y}(0,1+\varepsilon).$$

Let $Y_{\varepsilon}=(Y,\|.\|_{\varepsilon})$. By assumption $\mathcal{A}\left(X,Y_{\varepsilon}\right)$ is an ideal in $\mathcal{B}\left(X,Y_{\varepsilon}\right)$ so there exists a Hahn-Banach extension operator $Q_{\varepsilon}:\mathcal{A}\left(X,Y_{\varepsilon}\right)^{*}\rightarrow\mathcal{B}\left(X,Y_{\varepsilon}\right)^{*}$.Equivalently, the tensor space gives: $Q_{\varepsilon}\left(y^{*}\otimes x^{**}\right)=y^{*}\otimes x^{**}$ for all $y^{*}\in F_{\varepsilon}$, and $x^{**}\in X^{**}$. Let $I_{\varepsilon}:Y_{\varepsilon}\rightarrow Y$ denote the identity mapping. Then $\|I_{\varepsilon}{}^{-1}\|=1$ and $\|I_{\varepsilon}\|\rightarrow 1$ as $\varepsilon\rightarrow 0$. Define $\psi_{\varepsilon}\in\mathcal{H}\left(\mathcal{A}(X,Y)^{*},\mathcal{B}(X,Y)^{*}\right)$ by $\left(\psi_{\varepsilon}(\phi)\right)(T)=\left(Q_{\varepsilon}\left(\phi_{\varepsilon}\right)\right)\left(I_{\varepsilon}^{-1}\circ T\right),\phi\in\mathcal{A}(X,Y)^{*},T\in\mathcal{B}(X,Y)$, where $\phi_{\varepsilon}\in\mathcal{A}\left(X,Y_{s}\right)^{*}$ is defined by $\phi_{s}(S)=\phi\left(I_{s}\circ S\right),S\in(X,Y_{s})$ such that ψ is constructable. $\square$

Lemma 4.5.1. *Let* X *and* Y *be Banach spaces, and let* F *be a finite-dimensional subspace of* Y^{*}*. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be operator ideals satisfying* $\mathcal{A}\subseteq\mathcal{B}$*. The subset*

$$K_{F}=\{\psi\in H\ B(\mathcal{A}(X,Y)\ \mathcal{B}(X,Y)):\left(\psi\left(y^{*}\otimes x^{**}\right)\right)(T)=\left(y^{*}\otimes\right)x^{**}(T),$$

for all $y^{*}\in F,x^{**}\in X^{**}$*, and* $T\in\mathcal{B}(Y,X)\}$ *of* $\left(\mathcal{B}(X,Y)\widehat{\otimes}_{\pi}\mathcal{A}(X,Y)^{*}\right)^{*}$ *is compact in the weak* **-topology.*

Theorem 4.5.1. *Let* X *and* Y *be Banach spaces. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be operator ideals satisfying* $\mathcal{A}\subseteq\mathcal{B}$*. If* $\mathcal{A}(X,\hat{Y})$ *is an ideal in* $\mathcal{B}(X,\hat{Y})$ *for every equivalent renorming* $\hat{Y}$ *of* Y*, then there exists a* $\psi\in HB(\mathcal{A}(X,Y),\mathcal{B}(X,Y))$ *such that* $\left(\psi\left(y^{*}\otimes x^{**}\right)\right)(T)=\left(y^{*}\otimes x^{**}\right)(T)$ *for all* $y^{*}\in Y^{*}$*, all* $x^{**}\in X^{**}$ *and all* $T\in\mathcal{B}(X,Y)$*.*

Proof. Let K_{F} be as defined in Lemma 4.5.1 for every finite-dimensional subspace F of Y^{*} then each $K_{F}\neq\Phi$. If $F_{1},\ldots\ldots,F_{n}$ is a finite collection of finite-dimensional subspaces of Y^*, then $\cap_{i=1}^{n} K_{F_i} \neq \Phi$. Let $F = \operatorname{span}(F_1 \cup \ldots . \cup F_n)$. Then F is a finite dimensional subspace of Y^* and $\cap_{i=1}^{n} K_{F_i} \supseteq K_F \neq \Phi$. Now we know by compactness there is a $\psi \in \cap K_F, F \subseteq Y^*, \dim F < \infty$. For all $y^* \in Y^*$ there is a finite-dimensional subspace F of Y^* such that $y^* \in F$. Since $\psi \in K_F$ we have $(\psi(y^* \otimes x^{**}))(T) = (y^* \otimes x^{**})(T)$ for all $x^{**} \in X^{**}$ and $T \in \mathcal{B}(X, Y)$.

$\square$

We now state and prove our main result for dual spaces.

Theorem 4.5.2. *Let X be a Banach space. Given $\mathcal{F}(X, \hat{X})$ is an ideal in $\mathcal{H}(X, \hat{X})$ for every equivalent renorming $\hat{X}$ of X, then X has a shrinking metric approximation property.*

Proof. Starting with a Hahn-Banach extension operator $\psi : \mathcal{F}(X, X)^* \to \mathcal{H}(X, X)^*$ such that $\psi(x^* \otimes x^{**}) = x^* \otimes x^{**}$ for all $x^* \in X^*$ and $x^{**} \in X^{**}$. Theorem 5.2 in [10] now shows that X^* has the metric approximation property. $\square$

4.6 u-Ideals of Operators

Lemma 4.6.1. *Let X and Y be Banach spaces.Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $T \in \mathcal{B}(Y, X)$. If $\mathcal{A}(Y, X)$ is a u-deals in $\mathcal{B}(Y, X)$ and P is an ideal projection, then there exists a net $(T_\alpha) \subseteq \mathcal{B}(Y, X)$ with limsup $_\alpha \|T - 2T_\alpha\| \leq \|T\|$ such that $y^{**}(T_\alpha{}^* x^*) \to^\alpha (P(x^* \otimes y^{**}))(T)$ for all $x^* \in X^*$ and $y^{**} \in Y^{**}$.*

Theorem 4.6.1. *Let X be a Banach space and let Z be a separable subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $T \in \mathcal{B}(Y, X)$. If $\mathcal{A}(\hat{Y}, X)$ is a u-ideal in $\mathcal{B}(\hat{Y}, X)$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(Y, X), \mathcal{B}(Y, X)$) such that $(\psi(x^* \otimes z))(T) = (x^* \otimes z)(T)$ for all $x^* \in X^*$ and $z \in Z$ and $T \in \mathcal{B}(Y, X)$. Furthermore the ideal projection $P = \psi \circ i^*$, where i is a natural inclusion, satisfies $\|I - 2P\| = 1$.*

Proof. For every > 0, we can find an equivalent norm $\|.\|_\varepsilon$ on Y such that $Y_\varepsilon = (Y, \|.\|_\varepsilon)$ is locally uniformly rotund at every $z \neq 0$ in Z_ε and such that $B_Y(0,1) \subseteq B_{Y_\varepsilon}(0,1) \subseteq B_Y(0, 1+\varepsilon)$. Let $i_\varepsilon : \mathcal{A}(Y_\varepsilon, X) \to \mathcal{B}(Y_\varepsilon, X)$ be the identity map. By way of assumption $\mathcal{A}(Y_\varepsilon, X)$ is a u-deals in $\mathcal{B}(Y_\varepsilon, X)$.

So that there exists an ideal projection $\mathcal{G}_s$ such that $\|I - 2\mathcal{G}_s\| = 1$.We find a Hahn-Banach extension operator Q_ε such that $\mathcal{G}_\varepsilon = Q_\varepsilon \circ i^*$ and $(Q_\varepsilon (x^* \otimes z))(T) = (x^* \otimes z)(T)$ for all $x^* \in X^*$ and $z \in Z_\varepsilon$, and $T \in \mathcal{B}(Y_s, X)$. Let $I_\varepsilon : Y_\varepsilon \to Y$ denote the identity mapping. We define $\phi_\varepsilon \in \mathcal{A}(Y_\varepsilon, X)^*$ using $\phi \in \mathcal{A}(Y,X)^*$,and we define $\psi_\varepsilon \in \mathcal{H}(\mathcal{A}(Y,X)^*, \mathcal{A}(Y,X)^*)$ using Q_ε. Let $\mathcal{S} \in \mathcal{B}(Y,X)^*$ and define $\mathcal{S}^\varepsilon \in \mathcal{B}(Y_\varepsilon, X)^*$ by $\mathcal{S}^\varepsilon(T) = \mathcal{S}(T \circ I_\varepsilon^{-1}), T \in \mathcal{B}(Y_\varepsilon, X)$. For $T \in \mathcal{A}(Y_\varepsilon, X)$ and $\mathcal{S} \in \mathcal{B}(Y,X)^*$ we have $[i^*(\mathcal{S})]_\varepsilon (T) = i^*(\mathcal{S})(T \circ I_\varepsilon^{-1}) = \mathcal{S}(i(T \circ I_\varepsilon^{-1}))$

$$
\begin{aligned}
&= \mathcal{S}\left(i_\varepsilon(T) \circ I_\varepsilon^{-1}\right) = \mathcal{S}^\varepsilon(i_\varepsilon(T)) \\
&= i_\varepsilon^* \mathcal{S}^\varepsilon(T).
\end{aligned}
$$

So that these functionals have the same Q_ε extension. Let $P_\varepsilon = \psi_\varepsilon \circ i^*$. Then we have the following norm estimate:

$$
\begin{aligned}
&= \sup_{\mathcal{S} \in B_{\mathcal{B}(Y,X)^*}} \quad T \in B_{\mathcal{B}(Y,X)^-} \left|\mathcal{S}(T) - 2\left((Q_\varepsilon)[i^*(\mathcal{S})]_\varepsilon\right)(T \circ I_\varepsilon)\right| \\
&\leq \delta \in B_{\mathcal{B}(Y,X)^*} \|\mathcal{S}^\varepsilon - 2(Q_\varepsilon \circ i_\varepsilon{}^*)(\delta^\varepsilon)\|(1+\varepsilon) \\
&\leq \delta \in B_{\mathcal{B}(Y,X)^*} \|I - 2Q_\varepsilon\|(1+\varepsilon) \leq (1+\varepsilon).
\end{aligned}
$$

Since $\|I - 2Q_\varepsilon\| \leq 1$.

Since $(Q_\varepsilon)_{s \in (0,1]} \subseteq \mathcal{H}(\mathcal{A}(Y,X)^*, \mathcal{B}(Y,X)^*) = \left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$ is a bounded net it has a subnet $\left(Q_{\varepsilon(v)}\right)$ that converges weak * to some Q. In fact Q is a Hahn-Banach extension operator. Next we show that the projection P defined by $P = Q \circ i^*$ is the desired u-ideal projection.

Let $\mathcal{S} \in \mathcal{B}(Y,X)^*$ and $T \in \mathcal{B}(Y,X)$, then $P(\mathcal{S})(T) = Q\left(i^*(\mathcal{S})\right)(T) = \lim_v Q_{\varepsilon(v)}\left(i^*(\mathcal{S})\right)(T)$

$$= \lim{}_v P_{\varepsilon(v)}((\mathcal{S})(T) \text{ so that } \sup_{B(Y,X)} \mid \mathcal{S}(T) - 2P(\mathcal{S})(T) \mid$$

$$= s \in B_{\mathcal{B}(Y,X)^*} \quad T \in B_{\mathcal{B}(Y,X)} \quad \lim{}_v \mid \mathcal{S}(T) - 2P_{\varepsilon(v)}(\mathcal{S})(T) \mid$$

$$\leq s \in B_{\mathcal{B}(Y,X)^*} \quad T \in B_{\mathcal{B}(Y,X)} \quad \lim{}_v \|1 - 2P_{\varepsilon(v)}\| \|\mathcal{S}\| \|T\|$$

$$\leq \lim{}_v \left\|1 - 2P_{\varepsilon(v)}\right\| = 1$$

So we have as required $\|I - 2P\| = 1$. ☐

Lemma 4.6.2. *Let X be a Banach space and let Z be a separable subspace of a Banach space Y.Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $i : \mathcal{A} \to \mathcal{B}$ be the natural inclusion. The subset*

$$K_z = \left\{ \begin{array}{c} \psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y)) : \left(\psi\left(x^* \otimes z\right)\right)(T) = \left(x^* \otimes z\right)(T), \\ \forall x \in X^*, z \in Z \text{and } T \in \mathcal{B}(Y,X) \text{ and } \|1 - 2P\| = 1, \text{ where } P = Q \circ i^* \end{array} \right\}$$

is weak-compact in* $\left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$.

Proof. Let $\psi_\varepsilon \subseteq K_z$ be a net which converges weak * to some $\psi \in \left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$. Now we have $\left(\psi\left(x^* \otimes z\right)\right)(T) = \left(x^* \otimes z\right)(T)$ for all $x \in X^*, z \in Z$, and $T \in \mathcal{B}(Y,X)$. Let $P = Q \circ i^*$ and thus $\|1 - 2P\| \leq \lim_\alpha \|1 - 2P_\alpha\| = 1$. This shows that K_z is weak $^*-$ closed and since it is a bounded subnet it is weak $^*-$ compact. ☐

4.7 u-Ideals of Operators and Dual Spaces

Theorem 4.7.1. *Let X be a Banach space. Given a net $(T_\alpha) \subseteq \mathcal{F}(X,X)$ with $\limsup_\alpha \|1 - 2T_\alpha\| \leq 1$ such that $T_\alpha x \to x$ for all $x \in X$ and $T_\alpha x^* \to x^*$ for all $x^* \in X^*$, then $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{H}(X,Y)$ for every Banach space Y.*

Proof. We shall proof the case for finite rank operators. Let F be a finite dimension subspace of $\mathcal{H}(X,Y)$, and let $\varepsilon > 0$. Let $G = F \cap \mathcal{F}(X,Y)$. Then $K = \overline{U_{T \in B_G} T^*\left(B_{Y^*}\right)}$

is a compact subspace of X^*. By assumption we can find a $T \in \mathcal{F}(X,X)$ such that $\|1-2T\| \leq 1+\varepsilon$ and such that that $\|x^* - T^*x^*\| \leq \varepsilon$ for all $x^* \in K$. We define a linear map $L : F \to \mathcal{F}(X,Y)$ by $L(S) = ST$.Then $\|S - ST\| = \|S^* - T^*S^*\| < \varepsilon\|S\|$ for all $S \in G$ and $\|S - 2L(S)\| \leq (1+\ \varepsilon)\|S\|$ for all $S \in F$. By applying Proposition 3.6 in [4] we find that $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{H}(X,Y)$. $\square$

The following results are key to the proof of the second main result under this section.

Theorem 4.7.2. *Let X and Y be Banach spaces and let F be a finite dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X,\hat{Y})$ is a u-deals in $\mathcal{B}(X,\hat{Y})$ for every equivalent renorming $\widehat{Y}$ of Y, then there exists a*

$$\psi \in HB(\mathcal{A}(X,Y),\mathcal{B}(X,Y)) : \left(\psi\left(y^* \otimes x^{**}\right)\right)(T) = \left(y^* \otimes x^{**}\right)(T),$$

for all $y^* \in F, x^{**} \in X^{**}$, and $T \in \mathcal{B}(X,Y)$ Furthermore, the ideal projection $P = Q \circ i^*$ where i is a natural inclusion, satisfies $\|1-2P\| = 1$.

Lemma 4.7.1. *Let X and Y be Banach spaces and let F be a finite dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $i : \mathcal{A} \to \mathcal{B}$ be the natural inclusion. The subset*

$$K_Z = \psi \in HB(\mathcal{A}(X,Y),\mathcal{B}(X,Y)) : \psi\left(y^* \otimes x^{**}\right)(T) = \left(y^* \otimes x^{**}\right)(T), \forall y^* \in F, x^{**} \in X^{**}$$

$$T \in \mathcal{B}(Y,X)\ , \|1-2P\| = 1, P = Q \circ i^*$$

is weak *-compact in $\left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$.

Theorem 4.7.3. *Let X and Y be Banach spaces and let F be a finite dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X,\hat{Y})$ is a u-ideals in $\mathcal{B}(X,\hat{Y})$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(X,Y),\mathcal{B}(X,Y)) : \left(\psi\left(y^* \otimes x^{**}\right)\right)(T) = \left(y^* \otimes x^{**}\right)(T)$, for all $y^* \in Y^*, x^{**} \in X^{**}$, and $T \in \mathcal{B}(X,Y)$.*

Furthermore, the ideal projection $P = \psi \circ i^*$ where i is a natural inclusion, satisfies $\|1 - 2P\| = 1$.

We can now prove our next result under this section.

Proposition 4.7.1. *Let X be a Banach space. Given $\mathcal{F}(X, \hat{X})$ is a u-ideal in $\mathcal{H}(X, \hat{X})$ for every equivalent renorming $\hat{X}$ of X.Then there is a net $(T_\alpha) \subseteq \mathcal{F}(X, X)$ with* $\limsup_\alpha \|1 - 2T_\alpha\| \leq 1$ *such that $T_\alpha x \to x$ for all $x \in X$ and $T_\alpha x^* \to x^*$ for all $x^* \in X^*$.*

Proof. We limit ourselves to the case with finite rank operators. We find a Hahn-Banach extension operator

$$\psi : \mathcal{F}(X, X)^* \to \mathcal{H}(X, X)^*$$

such that $(\psi(x^* \otimes x^{**}))(T) = (x^* \otimes x^{**})(T)$ for all $x^* \in X^*, x^{**} \in X^{**}$ and $T \in \mathcal{H}(X, X)$. Let $i : \mathcal{F}(X, X) \to \mathcal{H}(X, X)$ be the natural inclusion, and let $P = \psi \circ i^*$ be the associated u-deal projection. Now there exists a net $(T_\alpha) \subseteq \mathcal{F}(X, X)$ with $\limsup_\alpha \|1 - 2T_\alpha\| \leq 1$ such that $x^{**}({T_\alpha}^* x^*) \to_\alpha (P(x^* \otimes x^{**}))(T)$ for all $x^* \in X^*$ and $x^{**} \in X^{**}$. Thus for all $x^* \in X^*$ and $x^{**} \in X^{**}$ $x^{**}({T_\alpha}^* x^*) \to {}_\alpha x^{**}(x^*)$, which means that ${T_\alpha}^* \to I^*$ in the weak operator topology on $\mathcal{H}(X^*, X^*)$. In particular we have $x^*(T_\alpha x) \to {}_\alpha x^*(x)$ for all $x^* \in X^*$ and $x \in X$, so that $T_\alpha \to I$ the weak operator topology on $\mathcal{H}(X, X)$. By choosing a new net in $\operatorname{conv}(T_\alpha)$, still denoted (T_α), we may assume ${T_\alpha}^* \to I^*$ in the strong operator topology on $\mathcal{H}(X^*, X^*)$ and $T_\alpha \to I$ in the strong operator topology on $\mathcal{H}(X, X)$.

□

4.8 λ-Bounded Approximation Properties in a Banach Space through Integral and Nuclear Operators

Let X and Y be Banach spaces. We denote by $L(X, Y)$ the Banach space of all bounded linear operators from X to Y, and by $\mathcal{F}(X, Y)$ and $\mathcal{W}(X, Y)$ its subspaces of finite rank operators and weakly compact operators. Let I_X denote the identity operator on X.

Definition 4.8.1. *We say X has the weak λ - bounded approximation property (weak $\lambda - BAP$) if for every Banach space Y and every operator $T \in \mathcal{W}(X, Y)$ there exists* *a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *such that* $S_\alpha \to I_X$ *uniformly on compact subsets of* X *and* $\limsup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$.

Thus the weak BAP can be characterized as the AP which is bounded for every weakly compact operator. This leads to the following definition

Definition 4.8.2. *Let Let* X *be a Banach space and let* $\mathcal{D} = (\mathcal{D}, \|\|_\mathcal{D})$ *be a Banach operator ideal. We say that* X *has the* λ *- bounded approximation property for* $\mathcal{D}$ *(weak* $\lambda - BAP$ *for* $\mathcal{D}$ *) if for every Banach space* Y *and evry operator* $T \in \mathcal{D}(X,Y)$ *there exists a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *such that* $S_\alpha \to I_X$ *uniformly on compact subsets of* X *and* $\limsup_\alpha \|TS_\alpha\|_\mathcal{D} \leq \lambda\|T\|_\mathcal{D}$.

For us to have a complete picture we recall other types of bounded approximation properties involving operator ideals which have been studied.

Definition 4.8.3. *Let* $\mathcal{D}$ *be an operator ideal. A Banach space* X *is said to have* λ *- bounded* $\mathcal{D}$ *- approximation property (*λ *- bounded* $\mathcal{D} - AP$*) if there exists a net* $(S_\alpha) \subset \mathcal{D}(X,X)$ *With* $\sup_\alpha \|S_\alpha\| \leq \lambda$ *such that* $S_\alpha \to I_X$ *uniformly on compact subsets of* X.

We establish reformulations of BAP in terms of the boundedness for the Banach operator ideals of strictly integral and integral operators respectively. The following Theorem and lemmas will be key in this direction.

Theorem 4.8.1. *Let* X *be a Banach space and I integral operator, and let* $1 \leq \lambda < \infty$. *The following statements are equivalent.*

(i) X *has the* $\lambda - BAP$.

(ii) $\|T\|_\pi \leq \lambda\|T\|_\mathrm{I}$ *for all* $T \in \mathcal{F}(X,X)$.

Lemma 4.8.1. *Let* X *be a Banach space, I an integral operator and* $1 \leq \lambda < \infty$. *If a Banach space* Y *has the property that for every* $T \in \mathrm{I}(X,Y^{**})$ *there exists a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *such that* $S_\alpha \to I_X$ *pointwise and* $\limsup_\alpha \|TS_\alpha\|_\pi \leq \lambda\|T\|_\mathrm{I}$, *then every quotient space of* Y *has the same property.*

Proof. Denote by $q : Y \to Z$ the quotient mapping, and let $U \in \mathrm{I}(X, Z^{**})$. Using well known results about tensor products, $\mathrm{I}(X, Z^{**}) = \left(Z^*\widehat{\otimes}_\varepsilon X\right)^*$ and $Z^*\widehat{\Phi}_\varepsilon X$ is a subspace of $Y^*\widehat{\otimes}_\varepsilon X$, we may consider a norm preserving extension of U. Thus there exists $T \in \left(Y^*\widehat{\otimes}_\varepsilon X\right)^* = \mathrm{I}(X, Y^{**})$, such that $\|T\|_{\mathrm{I}} = \|U\|_{\mathrm{I}}$ and

$$(Ux)(z^*) = \langle U, z^* \otimes x\rangle = \langle T, q^*z^* \otimes x\rangle = (Tx)(q^*z^*) = (q^{**}Tx)(z^*)$$

for all $x \in X$ and $z^* \in Z^*$, meaning that $U = q^{**}T$. Let $S \in \mathcal{F}(X, X)$. Then $US \in \mathcal{F}(X, Z^{**}) = X^* \otimes Z^{**}$ and $\|US\|_\pi = \|q^{**}TS\|_\pi = \|(I_X \otimes q^{**})(TS)\|_\pi \leq \|TS\|_\pi$.

Now if $(S_\alpha) \subset \mathcal{F}(X, X)$ is chosen for T, then we also have $\limsup_\alpha \|US_\alpha\|_\pi \leq \limsup_\alpha \|TS_\alpha\|_\pi \leq \lambda\|T\|_{\mathrm{I}} = \lambda\|U\|_{\mathrm{I}}$ □

Lemma 4.8.2. *Let X be a Banach space, and let $T \in \mathcal{F}(X, X) = X^* \otimes X$.Then there exists $A \in L(X^*, X^*)$ with $\|A\| = 1$ and $V \in \mathcal{F}(X, X)$ such that $V^* = AT^*$ and $\|T\|_\pi \leq \limsup_\alpha \|j_X VS_\alpha\|_\pi$ for every net $(S_\alpha) \subset \mathcal{F}(X, X)$ converging pointwise to the identity I_X.*

Proof. Let $T = \sum_{n=1}^m x^* \otimes X$. Using the canonical description $\left(X^*\widehat{8}_\pi X\right)^* = L(X^*, X^*)$, we find $A \in L(X^*, X^*)$ with $\|A\| = 1$ such that $\|T\|_\pi = \sum_{n=1}^m (Ax_n{}^*)(x_n) = \mathrm{trace}(V)$, where $V = \sum_{n=1}^m Ax_n{}^* \otimes x_n \in \mathcal{F}(X, X)$ it is easy to verify that $V^* = AT^*$.
Let $(S_\alpha) \subset \mathcal{F}(X, X)$ be a net such that $S_\alpha \to I_X$ pointwise. Since $X^*\widehat{\otimes}_\pi X$ is a subspace of $X^*\widehat{\otimes}_\pi X^{**}$ for all α, we have $\|VS_\alpha\|_\pi = \|j_X VS_\alpha\|_\pi$. Therefore

$$\begin{aligned}
\|T\|_\pi &= \sum_{n=1}^m (Ax_n{}^*)(x_n) = \lim_\alpha \sum_{n=1}^m (Ax_n{}^*)(S_\alpha x_n) \\
&= \lim_\alpha \sum_{n=1}^m (S_\alpha{}^* Ax_n{}^*)(S_\alpha x_n) \\
&= \mathrm{limtrace}(VS_\alpha) \leq \limsup_\alpha \|VS_\alpha\|_\pi = \limsup_\alpha \|j_X VS_\alpha\|_\pi .
\end{aligned}$$

□

Next, we proof the main result under this section.

Theorem 4.8.2. *Let X be a Banach space, I an integral operator and $1 \leq \lambda < \infty$.If X has λ-Bounded approximation property for I then it has λ-bounded approximation property.*

Proof. Since X has λ-Bounded approximation property for I, then for every $l_1(\Gamma)$-space and for every $T \in \mathrm{I}(\mathrm{X}, l_1(\Gamma)^{**})$ there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ pointwise and limsup $_\alpha \|TS_\alpha\|_1 \leq \lambda \|T\|_{\mathrm{I}}$.Since $TS_\alpha \in \mathcal{F}(X, l_1(\Gamma)^{**}) = X^* \otimes l_1(\Gamma)^{**}$ and $l_1(\Gamma)^{**}$ has the metric approximation preoperty, it is well known that $\|TS_\alpha\|_\pi = \|TS_\alpha\|_{\mathcal{N}} = \|TS_\alpha\|_I$. Recalling that every Banach space is a quotient of some $l_1(\Gamma)-$ space we may assume that for every $U \in \mathrm{I}(\mathrm{X}, X^{**})$ there exists (S_α) as above such that $\limsup_\alpha \|US_\alpha\|_\pi \leq \lambda \|U\|_{\mathrm{I}}$. Let $T \in \mathcal{F}(X, X)$. Choose A and V. Then choose a net $(S_a) \subset \mathcal{F}(X, X)$ to be pointwise convergent to I_X such that $\limsup \|j_X V S_\alpha\|_\pi \leq \lambda \|j_X V\|_I$. Then by $\|T\|_\pi \leq \lambda \|j_X V\|_I = \lambda \|V\|_I$.
On the other hand, since $V^* = AT^*$,

$$\|V\|_I = \|V^*\|_I = \|AT^*\|_I \leq \|T^*\|_I = \|T\|_I.$$

In conclusion,

$$\|T\|_\pi \leq \lambda \|T\|_I.$$

Which means that X has the λ-Bounded approximation property. $\square$

4.9 The Nuclear Operator and the Weak Bounded Approximation Property

Theorem 4.9.1. *Let X be a Banach space, and $1 \leq \lambda < \infty$. The following statements are equivalent.*

(a) X has the weak $\lambda - BAP$

(b) $\|T\|_\pi \leq \lambda \|j_X T\|_{\mathcal{N}}$, for all $T \in \mathcal{F}(X, X)$.

We will also need a reformation of the weak bounded approximation property in terms of extension operators as in the theorem that follows.

Definition 4.9.1. *Let X be closed subspace of a Banach space $W\cdot$ An operator $\Phi \in L(X^*, W^*)$ is called an extension operator if $(\Phi x^*)(x) = x^*(x)$ for all $x \in X$ and $x^* \in X^*$.*

Theorem 4.9.2. *Let X be a Banach space, and $1 \leq \lambda < \infty$. The following statements are equivalent.*

(a) X has the weak $\lambda - BAP$.

(b) There exists an extension operator $\Phi \in \overline{X \otimes X^*}^{w^*} \subset L(X^*, X^{***}) = \left(X^* \widehat{\otimes}_\pi X^{**}\right)^*$ with $\|\Phi\| \leq \lambda$.

Remark 4.9.1. *$AT \in X^* \widehat{\otimes}_\pi X^{**}$ is defined in the usual way if $T = \sum_n {x_n}^* \otimes u_n$, with ${x_n}^* \in X^*, u_n \in l_1$, then $AT = \sum_n {x_n}^* \otimes Au_n$.*

Theorem 4.9.3. *Let X be a Banach space, I an integral operator and $1 \leq \lambda < \infty$. The following statements are equivalent.*

(a) X has the weak $\lambda - BAP$

(b) X has the $\lambda - BAP$ for $\mathcal{N}$.

Proof. We first establish (b) for $Y = l_1$ since the nuclear operators factor through l_1 and the dual space of $\mathcal{N}(X, l_1) = X^* \widehat{\otimes}_\pi l_1$. Let Φ be the extension operator as defined in Theorem 4.9.2 and let $(S_v) \subset \mathcal{F}(X, X)$ be a net such that ${S_v}^* \to \Phi$ weak * in $L(X^*, X^{***}) = \left(X^* \widehat{\otimes}_\pi X^{**}\right)^*$. Let $T \in \mathcal{N}(X, l_1) = X^* \widehat{\otimes}_\pi l_1$. We may assume without loss of generality that $\|T\|_\pi = 1$. We show that every compact subset K of X and for every $0 < \varepsilon$ the convex subset $C = \{TS : S \in \mathcal{F}(X, X), \|Sx - x\| \leq \varepsilon \text{ for all } x \in K\}$ of $X^* \widehat{\otimes}_\pi l_1$ intersects the closed ball $B = \left\{u \in X^* \widehat{\otimes}_\pi l_1 : \|u\|_\pi \leq \lambda + \varepsilon\right\}$.
If these were not the case then, there would exist $A \in L(l_1, X^{**}) = \left(X^* \widehat{\otimes}_\pi l_1\right)^*$ with $\|A\| = 1$ such that

$$\begin{aligned}\lambda + \varepsilon = \sup\{\mathrm{Re}\langle A, u\rangle : u \in B\} &\leq \inf\{\mathrm{Re}\langle A, TS\rangle : TS \in C\} \\ &\leq \lim_v |\langle A, TS_v\rangle| = \lim_v |\langle {S_v}^*, AT\rangle| = |\langle \Phi, AT\rangle| \leq \lambda \|AT\|_\pi \leq \lambda\end{aligned}$$

a contradiction which establishes $Y = l_1$.

Next we let Y be a Banach space, let $T \in \mathcal{N}(X, Y)$ and let $\varepsilon > 0$. According to [5] there exists $R \in (l_1, Y)$ and $\hat{T} \in \mathcal{N}(X, l_1)$ with $\|R\| \leq 1$ and $\|\hat{T}\|_{\mathcal{N}} \leq \|T\|_{\mathcal{N}} + \varepsilon/\lambda$ such that $T = R\hat{T}$. Let $(S_\alpha) \subset \mathcal{F}(X, X)$ be a net such that $S_\alpha \to I_X$ uniformly on compact sets and next, we show (b) $\to$ (a).

Let $T \in \mathcal{F}(X, X)$. Choose A and V. Then $V^* = AT^*$ and for every net $(S_\alpha) \subset \mathcal{F}(X, X)$ converging pointwise to I_X. Since $j_X V \in \mathcal{N}(X, X^{**})$, for every $\varepsilon > 0$, we can write $j_X V = \sum_{n=1}^{\infty} x_n{}^* \otimes x_n{}^{**}, x_n{}^* \in X^*, x_n{}^{**} \in X^{**}$, With $\sum_{n=1}^{\infty} \|x_n{}^*\| \, \|x_n{}^{**}\| < \|j_X V\|_{\mathcal{N}} + \varepsilon$.

Now choose an $l_1(\Gamma)$-space such that X is its quotient space and denote $q : l_1(\Gamma) \to X$ the quotient mapping. q^* will be an isometric embedding hence for all $x_n{}^{**}$, there exists $u_n{}^{**} \in l_1(\Gamma)^{**}$ such that $q^{**} u_n{}^{**} = x_n{}^{**}$ and $\|u_n{}^{**}\| = \|x_n{}^{**}\|$.

Define and choose a net $(S_\alpha) \subset \mathcal{F}(X, X)$ converging pointwise to I_X such that

$$\limsup_{\alpha} \|US_\alpha\|_{\mathcal{N}} \leq \lambda \|U\|_{\mathcal{N}} \leq \lambda \sum_{n=1}^{\infty} \|x_n{}^*\| \, \|u_n{}^{**}\| \, .$$

Moreover,

$$\|j_X V\|_{\mathcal{N}} = \|V^*\|_{\mathcal{N}} = \|AT^*\|_{\mathcal{N}} \leq \|T^*\|_{\mathcal{N}} = \|j_X T\|_{\mathcal{N}} \, .$$

On the other hand, it can be easily verified that $j_X V = q^{**} U$. Hence $j_X V S_\alpha = q^{**} U S_\alpha$. Therefore,

$$\|j_X V S_\alpha\|_\pi = \|q^{**} U S_\alpha\|_\pi \leq \|US_\alpha\|_\pi = \|US_\alpha\|_{\mathcal{N}} \, .$$

In conclusion

$$\|T\|_\pi \leq \limsup_{\alpha} \|j_X V S_\alpha\|_\pi \leq \limsup_{\alpha} \|US_\alpha\|_{\mathcal{N}} < \lambda \, \|j_X T\|_{\mathcal{N}} + \varepsilon.$$

By letting $\varepsilon \to 0$, we have $\|T\|_\pi < \lambda \, \|j_X T\|_{\mathcal{N}}$ which means that X has weak Bounded approximation property as required. $\square$

CHAPTER FIVE

MODULES, TENSORS AND FUNCTORS IN BANACH ALGEBRA OVER FRECHET SPACES

5.1 Introduction

Let $\mathbb{A}$ be a Banach Algebra. In [59, 60, 61] Rieffel made an elaborate study of the Banach module $Hom_{\mathbb{A}}(\mathbb{A}, X)$ of continuous homomorphisms. Further results in this direction have been obtained by Sentilles and Taylor[68] and Ruess[66] in their study of the general strict topology. Related studies are attributed to among others, Shantha[71] whose study focussed on homomorphisms in the case of locally convex modules.

An algebra $\mathbb{A}$ (over $\mathbb{K}$) with a topology τ is called a topological algebra if it is a topological vector space (TVS) commonly called the Frechet Space, in which multiplication is separately continuous. A complete metrizable topological algebra is called an $F-$algebra; in this case the multiplication is jointly continuous by Arens' Theorem [[49], p. 24]. A net $\{e_\alpha : \alpha \in I\}$ in a topological algebra $\mathbb{A}$ is called a left approximate identity (respectively right approximate identity, two-sided approximate identity) if, for all $a \in \mathbb{A}$, $lim_\alpha e_\alpha a = a$ (respectively $lim_\alpha a e_\alpha = a$, that is $lim_\alpha e_\alpha a = lim_\alpha a e_\alpha$, $\{e_\alpha : \alpha \in I\}$ is said to be uniformly bounded if there exists $r > 0$ such that $\{(\frac{e_\alpha}{r})^n : \alpha \in I;\ n = 1, 2, \cdots\}$ is a bounded set in $\mathbb{A}$. A TVS (E, τ) is called ultrabarrelled if any linear topology τ' on E, having a base of neighbourhoods of 0 formed of $\tau-$closed sets, is weaker than τ. The Frechet space (E, τ) is called ultrabornological [31] if every linear map from E into any TVS which takes bounded sets into bounded sets is continuous. Every Baire TVS (in particular, $F-$space) is ultrabarrelled. Every metrizable TVS is ultrabornological.

Let X be a Frechet space and $\mathbb{A}$ be a topological algebra, both over the same field $\mathbb{K}$. Then X is called a topological left $\mathbb{A}-$module if it is a left $\mathbb{A}-$module and the module multiplication $(a,x) \rightarrow a.x$ from $\mathbb{A} \times X$ into X is separately continuous. If $b(\mathbb{A})$ (respectively $b(X)$) denote the collection of all bounded sets in $\mathbb{A}$ (respectively X), then module multiplication given above is called $b(\mathbb{A})$-hypocontinuous (respectively $b(X)$ hypocontinuous)[49] if, given any neighbourhood G of 0 in X and any $D \in b(\mathbb{A})$ (respectively $B \in b(X)$), there exists a neighbourhood H of 0 in X (respectively V of 0 in $\mathbb{A}$) such that $D.H \in G$ (respectively $V.B \in G$). Clearly, joint continuity implies hypocontinuity which also implies separate continuity; however, the converse need not hold. If E and X are TVSs, BL(E,X) (respectively CL(E,X)) denotes the vector space of all bounded (respectively continuous) linear mappings from E into X. Clearly, $CL(E,X) \in BL(E,X)$ with $CL(E,X) = BL(E,X)$ if E is ultrabornological (in particular metrizable). A mapping $T : E \rightarrow X$ is called a topological isomorphism if T is linear and a homeomorphism. If X is a left $\mathbb{A}-$module, then $\mathbb{A}$ is said to be faithful in X if, for any $x \in X,\ a.x = 0$ for all $a \in \mathbb{A}$ implies that $x = 0$ (cf. [35],[68]).

In the sequel, we begin with the fundamentals of Banach Rings and Modules in generalized forms:

5.2 Banach Rings and Banach Modules

Definition 5.2.1. *[21] A commutative ring R with identity is called a Banach Ring if it is equipped with a function* $|\cdot| : R \longrightarrow \mathbb{R}_{\geq 0}$ *such that:*

i. $|a| = 0 \ iff \ a = 0$

ii. $|a+b| \geq |a| + |b| \ \forall a, b \in R$

iii. $\exists a \ \delta \ > 0$ *such that* $|ab| \leq \delta |a||b| \ \forall a, b \in R$

iv. R is a complete metric space with respect to the metric $(a,b) \longrightarrow |a-b|$.

The category of Banach Rings has morphisms $\phi : R \rightarrow S$ such that there exists constant $\delta > 0$ with $|\phi(a)|_s \leq \delta \ |a|_R$. These are the bounded ring homomorphisms.

Definition 5.2.2. *Let $(R, |\cdot|_R)$ be a Banach Ring. A Banach module over R is an $R-$Module M equipped with a function $\|\cdot\|_M : M \to \mathbb{R}_{\geq 0}$ such that for any $m, n \in M$ and $a \in R$:*

i. $\|0_M\| = 0$

ii. $\|m+n\|_M \leqslant \|m\|_M + \|n\|_M$

iii. $\|am\|_M \leqslant | a_R \|_m\|_M$

iv. $\|m\|_m = 0 \Leftrightarrow m = 0_M$

v. M is complete with respect to the metric $d(m, n) = \|m - n\|$

Example:

Any Abelian Group or Ring can be considered as a Banach Ring by equiping it with the trivial norm which assign it zero and 1 for each non-zero element. For instance

$$(\mathbb{Z}_{triv} = (\mathbb{Z} \mid \cdot \mid); |a| = \begin{cases} 0 : a = 0 \\ 1 : \text{ otherwise} \end{cases} \qquad : a \in \mathbb{Z}$$

Let M be a module over a Banach Ring R, then M can be transformed to a Banach module if it is equipped with the trivial norm.

Definition 5.2.3. *[76]*

A Banach Ring or a Banach Module over a Banach Ring is called non-Archimedean if its semi-norm obeys the strong triangle inequality: for any two elements $v, w \in M$,

$$\|v + w\|_M \leqslant \max\{\|v\|, \|w\|\}.$$

Now, if $(M, \|\cdot\|_M)$ is a module over a Banach ring R and r is a positive real number, then M_r is the Banach module over R defined by the underlying module M equipped with the Banach structure $\|r\|_M$.

Definition 5.2.4. *[76] Let $(R, |\cdot|_R)$ be a Banach Ring. A R-linear map between Banach $R-$ module $f : (M, \|\cdot\|_M) \to (N, \|\cdot\|_N)$ is called bonded if there exists a real constant $\delta > 0$ such that: $\|f(m)\|_N \leq \delta\|m\|_M$ for any $m \in M$. The homomorphism is called non-expanding if the equation holds for $\delta = 1$.*
In the sequel, the category of Banach Module with bounded morphisms is denoted by Ban_R. If R is a non-archimedean $\overline{B}an_R$ denoted the category of non-archimedean Banach Modules with bounded morphisms.

The natural characteristics to study involve flatness, projective properties, tensor products and invertibility.

Lemma 5.2.1. *For any Banach ring R, R is projective as a Banach $R-$module.*

Lemma 5.2.2. *For any projective $R-$module P and any real number $r > 0$, P_r is also projective.*

Definition 5.2.5. *Given $M, N \in Ban_R$, we define $M\hat{\otimes}_R N$ as the separated completion of $M \otimes_R N$ with respect to the semi-norm:*

$$\|x\| = \inf\left\{\sum_{i=1}^{n} \|m_i\|\,\|n_i\| : x = \sum_{i=1}^{n} m_i \hat{\otimes}_R n_i\right\}.$$

Dually, if R is non-Archimedean and $M, N \in \overline{B}an_R$, then $\overline{M\hat{\otimes}_R N}$ is the separated completion of $M \otimes_R N$ with respect to the semi- norm:

$$\|x\| = \inf\left\{\text{Sup}\,\|m_i\|\,\|n_i\| : x = \sum_{i=1}^{n} \overline{m_i \hat{\otimes} n_i} : i = 1, \cdots, n\right\}.$$

The internal homomorphism in these categories is denoted by $\underline{Hom}_R(V, W)$ and given by the Banach space whose underlying vector space is the bounded R-linear maps.

$$\{T \in Lin_R(V, W) : \|T\| < \infty\}$$

with norm given by:

$$\|T\| = \sup \frac{\|T(V)\|}{\|V\|}; v \in V \text{ and } v \neq 0$$

The categories Ban_R and $\overline{B}an_R$ are both closed, symmetric monoidal when equipped with these projective tensor product with unit object by R.

Definition 5.2.6. *The category $Ban^{\leq 1}{}_R$ may be defined to have the same objects as Ban_R. The morphisms here are the linear maps with norm less than or equal to 1: the non expanding or contracting case.*

This defines a closed monoidal symmetric category, with the same internal hom and tensor product and as before having two versions (one of which exists when R is non Archimedean).

The internal products and coproducts in the infinite case exist in $Ban^{\leq 1}{}_R$. Indeed, in the Archimedean case, the product is given by:

$$\prod_{i\in I}^{\leq 1} V_i = \left\{ (v_i) \in \times_{i\in I} V_i : \sup_{i\in I} \|v_i\| < \infty \right\}$$

$$\text{where } \{V_i\} \in \}_{i\in I} \in \mathrm{Ban}^{\leq 1}{}_{Ri}$$

$$\|(v_i)\| = \sup_{i\in I} \|v_i\| .$$

On the other hand, the co-product $\coprod_{i\in I}^{\leq 1} V_i$ of a collection $\{V_i\}_{i\in I} \in Ban_R^{\leq 1}$ is:

$$\coprod_{i\in I}^{\leq 1} V_i = \left\{ (v)_{i\in I} \in \times_{i\in I} V_i : \sum_{i\in I} \|v_i\| < \infty \right\}$$

equipped with the norm:

$$\left\|(v_i)_{i\in I}\right\| = \sum_{i\in I} \|v_i\| .$$

In the non-Archimedean case, the above products are given by:

$$\prod_{i\in I}^{\leq 1} V_i = \left\{ (v_i) \in \times_{i\in I} V_i : \sup_{i\in I} \|v_i\| < \infty \right\}$$

equipped with the norm

$$\|(v_i)\| = \sup_{i\in I} \|v_i\| .$$

While the co-product is:

$$\coprod_{i\in I}^{\leq 1} V_i = \left\{ (v_i) \in \times_{i\in I} V_i : \lim_{i\in I} \|v_i\| = 0 \right\}$$

equipped with the norm

$$\|(v_i)\| = \sup_{i\in I} \|v_i\| .$$

The general limits and co-limits are constructed out of kernels and products respective co-kernels and co-products in the usual way.

ln fact, finite limits and finite co-limits in $Ban_R^{\leq 1}$ agrees with those in Ban_R.

Lemma 5.2.3. *Let $\{f_i : V_i \to W_i\}_{i\in I}$ be a collection in $Ban_R^{\leq 1}$. Then the natural map*

$$\coprod_{i\in I}^{\leq 1} \ker(f_i) \to \ker\left[\coprod_{i\in I}^{\leq 1} V_i \to \coprod_{i\in I}^{\leq 1} W_i\right]$$

is an isomorphism. Moreover, if $V_i \subset V$ and $W_i \subset W$ are countable increasing unions complete closed isometric sub-modules with union V and W respectively, then the natural map:

$$\text{collin}_{i\in I}^{\leq 1} \ker(f_i) \longrightarrow \text{Ker}[V \to W]$$

is an isomorphism.

Proposition 5.2.1. *A module M of Ban_R is projective if and only if there exists a set S and a map $f : S \to \mathbb{R}$ and another module N along with an isomorphism*

$$M \coprod N \cong \coprod_{s\in S}^{\leq 1} R_{f(s)}.$$

Proof. Using the method in [21], there exits a canonical strict epimorphism from

$$\coprod_{m\in M^*}^{\leq 1} R_{\|m\|} \to M$$

so if M is projective, this splits.

Conversely, if $M\coprod N \cong \coprod_{s\in S}^{\leq 1} R_{f(s)}$ and there exist sub-modules F and E and $F \to E$ is a strict epimorphism then:

$$\mathrm{Hom}(\coprod_{S\in S}^{\leq 1} R_{f(S)}) \to \mathrm{Hom}(\coprod_{S\in S}^{\leq 1} R_{f(S)}, E)$$

is a surjection and this breaks up into a product of a map Hom $(M, F) \to \mathrm{Hom}(M, E)$ and a map $Hom(M, F) \longrightarrow Hom(M, F)$ and a $Hom(N, F) \longrightarrow Hom(N, E)$ and so, these are both surjective. Hence M is projective. □

Proposition 5.2.2. *Let M and N be projective in Ban_R, then $M\hat{\otimes}_R$ is also a projective in Ban_R.*

Proof. Using the previous result, we can complement the modules, M and N with modules M' and N' such that there is a module:

$$S = \left(M\hat{\otimes}_R N'\right) \oplus \left(M'\hat{\otimes}_R N\right) \oplus \left(M'\hat{\otimes}_R N'\right).$$

So that

$$\left(M\hat{\otimes}_R N\right) \oplus S \cong \coprod_{(m,n)\in M^*\times N^*}^{\leq 1} R_{\|m\|\|n\|}$$

and the result follows. □

Definition 5.2.7. *A finitely generated R-Module $L \in Ban_R$ is said be invertible if there exists a prime ideal $P\triangle R$ such that there corresponds a localization morphism $L_p \cong R_p$. So L is locally free of rank* 1.

Rank $L = \dim\left(K \otimes_R L\right)$ *where K is a Banach Algebra.*

If $M, L \in Ban_R$ are invetible then

$$(L \otimes_R M)_P \cong L_P \otimes_{R_P} M_P \text{ and}$$
$$\mathrm{Hom}_R(L_P, R_P)_P \cong \mathrm{Hom}_{R_P}(L, R_p).$$

This means that $(L \otimes_R M)$ and

$$L^\vee = \mathrm{Hom}_R(L, R) \text{ are invertible too .}$$

We get the following new result:

Theorem 5.2.1. *Let L be an R-module over a Banach ring R and let the evaluation map $\delta : L \otimes_R L^V \to R$ defined by $\delta(x \otimes \lambda) = \lambda(x)$ be an isomorphism in Ban_R. Then L is invertible.*

Proof. Let L be invertible, then there exists an evaluation map: guaranteeing an ideal $P \Delta R : \delta_p : L_p \otimes_{R_p} L_p{}^\vee \approx R_p \otimes_{R_p} R_p \longrightarrow R_p$ is an isomorphism.
It then follows that δ is an isomorphism.

Conversely, let δ be an isomorphism and let $y_1, \ldots, y_n \in L$ and $\lambda_1, \ldots, \lambda_n \in L^\vee$ such that $\sum_{i=1}^n \lambda_i (y_i) = 1$.
Algebraically, if P is a prime ideal and $\lambda_i (y_i) \in P; i = 1, \ldots, n$ then $1 \in P$, a contradiction.
Let $\lambda_1(y_1) \notin P$ and hence $\lambda_1(y_1)$ is a unit in R_p.
Let $z = [\lambda_1(y_1)]^{-1} y_1 \in L_p$. Then the map $\lambda_1 : L_p \to R_p$ is surjective.
Since $\lambda_1(z) = 1$ and thus splits R_p (a free module).
Since $\delta : L \otimes_R L^\vee \to R$ is an isomorphism $\delta_p : L_p \otimes_{R_p} L_p^\vee \to R_p$ is an isomorphism.
Now if z is viewed as a map, whose kernel is zero, we use the decomposition:

$$L_p \otimes_{R_p} L_p^\vee \approx [zR_p \otimes \lambda_1 R_p] \oplus [zR_p \otimes \ker(z)] \oplus [\ker(\lambda_1) \otimes \ker(z)]$$

$\Longrightarrow \; L_p \cong zR_p \quad \text{and} \quad L_p^\vee \cong \lambda_1 R_p.$

Finally, if $M \subseteq L$ is a sub-module generated by $y_1 \cdots y_n$ and $f : M \to L$ is an inclusion map so that f_p is an isomorphism, then $M = L$ so that L is finitely generated. $\square$

Theorem 5.2.2. *Let M and N be projective in* Ban_R*. Then $M \hat{\otimes} N$ is flat in* Ban_R*.*

Proof. Formally a module M over Ban_R is flat if taking the tensor product over R with M preserves the exact sequences.
Now, consider the sequence of modules and map f, g given by

$$0 \xrightarrow{f} M \otimes N \xrightarrow{g} 0.$$

The sequence is short and exact since $Im(f) = ker(g)$.
Let $N = R$ and M' be projective in Ban_R then there exists a canonical strict isomorphism say $f : \coprod_{m \in M^*}^{\leq 1} \longrightarrow M$ which splits so that the coproduct $\coprod_{m \in M^*} R_{\|m\|}$ is a coproduct in Ban_R of kernel M. Hence M is a flat.

$\square$

The following lemma shall be useful in the next result.

Lemma 5.2.4. *Let R be a Banach Ring and M a Banach $R-$ module. Then for any positive real number r, we have:*

(i.) $(M_r)^{\vee} \cong (M^r)_{r-1}$.

(ii.) M_r is projective if and only if M is projective.

Theorem 5.2.3. *Given an inductive system V_i in $Ban_R^{\leq 1}$, the canonical morphism*

$$\left(\mathrm{Colim}_{i \in I}^{\leq 1} V_i\right)^{\vee} \to \lim_{i \in I}^{\leq 1} (V_i^{\vee})$$

induced by the duals of collections of isometric immersions

$$V_i \to co \lim_{i \in I} V_i$$

is an isomorphism.

Proof. It is enough to show that it induces an isomorphism of sets:

$$\left(\left(\left(\operatorname{colim}^{\leq 1}_{i\in I} V_i\right)^{\vee}\right)^{\leq r} \longrightarrow \lim_{i\in I}^{\leq 1} \left(V_i^{\vee}\right)\right)^{\leq r}$$

for any real number $r \geq 1$

The canonical morphism identifies the left handside with:

$$\begin{aligned}
\operatorname{Hom}^{\leq 1}\left(R_r\left(\operatorname{colim}^{\leq 1}_{i\in 1} V_i^v\right)\right) &= \operatorname{Hom}^{\leq 1}\left(R_{r,}\ \underline{\operatorname{Hom}}\ \left(\operatorname{colim}^{\leq 1}_{i\in I} V_i, R\right)\right). \\
&= \operatorname{Hom}^{\leq 1}\left(R_r \hat{\otimes}_R \left(\operatorname{Colim}^{\leq 1}_{i\in I} V_i\right), R\right). \\
&= \operatorname{Hom}^{\leq 1}\left(\operatorname{Colim}^{\leq 1}_{i\in I}\left(\left(V_i\right)_r\right), R\right). \\
&= \lim_{i\in I} Hom^{\leq 1}\left(\left(V_i\right)_r, R\right). \\
&= \lim_{i\in I} Hom^{\leq 1}\left(R, \left(\left(V_1\right)_r\right)^{\vee}\right) \\
&= \lim_{i\in I} Hom^{\leq 1}\left(R, \left(V_i^v\right)_{r=1}\right. \\
&= \lim_{i\in I} Hom^{\leq 1}\left(R_r, \lim_{i\in I}^{\leq 1}\left(V_i^{\vee}\right)\right)
\end{aligned}$$

which agrees with the right hand side. ∎

In the sequel, we consider Hausdorff algebras and modules over vector valued Topological spaces.

The following result follows from[19], but we include its proof.

Lemma 5.2.5. *Let (X,τ) be a topological left $\mathbb{A}-$module. If X is ultrabarrelled, then the module multiplication is $b(\mathbb{A})-$hypocontinuous.*

Proof. Let G be a neighbourhood of 0 in X and $D \in b(\mathbb{A})$. For any $a \in \mathbb{A}$, define $L_a : X \to X$ by $L_a(x) = a \cdot x,\ a \in \mathbb{A}$. Clearly, each L_a is linear and also continuous (by separate continuity of the module multiplication). Further, $\{L_a : a \in D\}$ is pointwise bounded in $CL(X, X)$. [Let $x \in X$ and G_1 a neighbourhood of 0 in X. Since D is bounded in $\mathbb{A}$, by separate continuity of the module multiplication, $D.x$ is bounded in X and so there exists $r > 0$ such that $D \cdot x \subseteq rG_1$. So,

$$\{L_a(x)\} : a \in D\} = \{a \cdot x : a \in D\} = D.x \subseteq rG_1,$$

showing that $\{L_a : a \in D\}$ is pointwise bounded in $CL(X, X)$.] Since X is ultrabarralled, by the principle of uniform boundedness [19], $\{\mathrm{L}_a : a \in D\}$ is equicontinuous. Hence, given any neighbourhood G of 0 in X, there exists a neighbourhood H of 0 in X such that $L_a(H) \subseteq G$ for all $a \in \mathbb{A}$; i.e. $D \cdot H \subseteq G$ $\square$

If (X, τ) is a topological left $\mathbb{A}$-module with $\mathbb{A}$ having a left approximate identity $\{e_\lambda : \lambda \in I\}$, the essential part X_e of X is defined as:

$$X_e = \{x \in X : e_{\mathbb{A}} \cdot X \longrightarrow^{\tau} x\}.$$

Clearly, $\mathbb{A}.X \subseteq X_e$ and X_e is a topological left $\mathbb{A}$-submodule of X. We say that X is essential if $\mathrm{X} = \mathrm{X_e}$.

We state the following theorem:

Theorem 5.2.4. *Let* (X, τ) *be a topological left* $\mathbb{A}$*-module with* X *ultrabarralled and* $\mathbb{A}$ *having a bounded left approximate identity* $\{e_\lambda : \lambda \in \mathrm{I}\}$. *Then* $\mathrm{X_e}$ *is* τ*-closed in* X.

Proof. Let $x \in \tau - cl\,(X_e)$. We need to show that $e_\lambda \cdot x \longrightarrow^{\tau} x$. Let G be a neighbourhood of 0 in X. Choose a balanced neighbourhood H of 0 in X such that $H + H + H \subseteq G$. For each $\lambda \in I$, define $L_\lambda : X \to X$ by $L_\lambda(y) = L_{e\lambda}(y) = e_\lambda : x, y \in X$. Since $D = \{e_\lambda : \lambda \in I\}$ is bounded in $\mathbb{A}$, it follows that $\{L_\lambda : \lambda \in I\}$ is pointwise

$$L_\lambda\,(H_1) \subseteq H \quad \text{for all } \lambda \in \mathrm{I}.$$

Since $x \in \tau - cl\,(X_e)$, we can choose $x_0 \in X_e$ such that

$$x - x_0 \in H_1 \cap H.$$

Since $e_\lambda x_0 \to x_0$ and so there exists $\lambda_2 \in I$ such that $e_\lambda - x_0 \in H \quad \forall \lambda \geq \lambda_0$. Hence,

$$e_\lambda x - x = e_\lambda \cdot (x - x_0) + (e_\lambda x_0 - x_0) + (x_0 - x) \in L_\lambda\,(H_1 \cap H) + H + H_1 \cap H \subseteq H + H + H \subseteq G$$

that is, $e_\lambda \cdot x \to \tau x$ and so $x \in X_e$. Hence X_e is $\tau-$closed.

□

We now state a generalization of a factorization Theorem which will be used later in the sequel.

Recall that, a topological algebra $\mathbb{A}$ is called *strongly factorable* if, for any sequence $\{a_n\}$ in $\mathbb{A}$ with $c_n \to 0$, there exist $b \in \mathbb{A}$ and a sequence $\{c_n\}$ in $\mathbb{A}$ with $c_n \to 0$ such that $a_n = c_n b$ for all $n \geqslant 1$ [5].

Theorem 5.2.5 (Cohen's Factorization Theorem)**.** *Let $\mathbb{A}$ be a fundamental* F*-algebra with a uniformly bounded left approximate identity. Then:*

i. $\mathbb{A}$ is strongly factorable.

ii. If X *is an* F*-space which is an essential topological left* A*-module, then* X *is $\mathbb{A}$-factorable.*

We mention that if X is $\mathbb{A}$-factorable, then X is essential since $X = \mathbb{A}.X \subseteq X_e \subseteq X$, or that $X = X_e$.

Definition 5.2.8. *[59] Let E and X be topological left $\mathbb{A}$-modules, where E and X are TVSs and $\mathbb{A}$ is a topological algebra, then a mapping $T : \mathrm{E} \to \mathrm{X}$ is called an $\mathbb{A}-$module homomorphism if $T(a \cdot x) = \mathrm{a} \cdot T(x)$ for all $a \in \mathbb{A}$ and $x \in E$.*

Now, a module homomorphism is not assumed to be linear or continuous. Our main interest here is the study of $\mathbb{A}-$module homomorphisms from $\mathbb{A}$ into X. The following algebraic result is an extension of the findings in[35].

Lemma 5.2.6. *Let* X *be a left* $\mathbb{A}$*-module. Suppose that* $\mathbb{A}$ *is faithful in* X*. Then any* $\mathbb{A}$*-module homomorphism* $T \to \mathrm{X}$ *is homogeneous (that is,* $\mathrm{T}(\lambda \mathrm{a}) = \lambda \mathrm{T}(\mathrm{a})$ *for all* $\lambda \in \mathrm{K}$ *and* $\mathrm{a} \in \mathbb{A}$*).*

Proof. Let $a \in \mathbb{A}$ and $\lambda \in K$. Then, for any $c \in \mathbb{A}$,

$$c.T(\lambda a) = T(c.(\lambda a)) = T((\lambda c)a) = (\lambda c).T(a) = c.\lambda T(a).$$

Since $\mathbb{A}$ is faithful in $X, T(\lambda a) = \lambda T(a)$. □

Next, we establish the linearity and continuity of an $\mathbb{A}-$module homomorphisms using the factorization theorem.
The following theorem extends some results in [33, 38, 59, 73] to our more general setting.

Theorem 5.2.6. *Let* X *be a topological left* $\mathbb{A}$*-module with* X *metrizable and* $\mathbb{A}$ *strongly factorable. Then any* $\mathbb{A}-$ *module homomorphism* $T\mathbb{A} \to \mathrm{X}$ *is linear and continuous.*

Proof. To show that T is linear, let $a_{1,}a_2 \in \mathbb{A}$ and $\alpha, \beta \in K$. If we take $\{a_n\} = \{a_1, a_2, 0, 0, \ldots\}$, then clearly $a_n \to 0$; since $\mathbb{A}$ is strongly factorable, there exist $b, c_1, c_2 \in \mathbb{A}$ such that $a_1 = c_1 b, a_2 = c_2 b$. So

$$\begin{aligned} &T(\alpha a_1 + \beta a_2) = T((\alpha c_1 + \beta c_2) b) \\ &= (\alpha c_1 + \beta c_2).T(b) = \alpha T(c_1 b) + \beta T(c_2 b) = \alpha T(a_1) + \beta(a_2). \end{aligned}$$

hence T is linear. Since X is metrizable, to show that T is continuous, it suffices to show that if $\{a_n\} \subseteq \mathbb{A}$ with $a_n \to 0$, then $T(a_n) \to 0$. Using again the strong factorability of $\mathbb{A}$, we can write $a_n = c_n b$, where $b \in \mathbb{A}$ and $\{c_n\} \subseteq \mathbb{A}$ with $c_p \to 0$. Then

$$T(a_n) = T(c_n b) = c_n \cdot T(b) \to 0.T(b) = 0$$

so by the separate continuity of module multiplication. Thus T is continuous.

□

5.3 Frechet Module $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$

Definition 5.3.1. *[59] Let E and X be topological left $\mathbb{A}$-modules, where E and X are TVSs and $\mathbb{A}$ is a E into X. If E is an $\mathbb{A}$-bimodule, then defining $(a * T)(x) = T(x\ .a)$, then,* $\mathrm{Hom}_{\mathbb{A}}(E, X)$ *becomes a left $\mathbb{A}$-module. In fact, for any $b \in \mathbb{A}, x \in E$*

$$(a * T)(b \cdot x) = T((b \cdot x) \cdot a) = T(b \cdot (x.a)) = b \cdot T(x.a) = b \cdot (a * T)(x).$$

In particular, $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$ is a left $\mathbb{A}$-module. Note that if A is commutative, then defining $(T*)(x) = T(a \cdot x), \mathrm{Hom}\,\mathbb{A}(E, X)$ becomes a right $\mathbb{A}$-module.

The structures of $\mathrm{Hom}_{\mathbb{A}}(E, X)$ have been determined in the case of E and X as the Banach modules of Banach valued function spaces $L^1(G, \mathbb{A})$ and $C_\circ(G, \mathbb{A})$, where G is a locally compact Abelian group and $\mathbb{A}$ is a commutative Banach algebra. Abel [1] studied it in the setting of topological bimodule algebras.

If $E = X = \mathbb{A}$, then $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A})$ is the usual multiplier algebra of $\mathbb{A}$, and is denoted by $M(\mathbb{A})$. In fact, there is a vast literature dealing with the notions of left multiplier, right multiplier, multiplier and double multiplier (see, e.g., [11, 29, 33, 35, 38, 56, 76]).

Lemma 5.3.1. *Let* E *and* X *be topological left $\mathbb{A}$-modules with $\mathbb{A}$ having an approximate identity $\{e$: $\lambda \in I\}$. If* E *is an essential $\mathbb{A}-$ module, then*
$\mathrm{Hom}_{\mathbb{A}}(E, X) = \mathrm{Hom}_{\mathbb{A}}\,(\mathrm{E}, \mathrm{X_e})$. *In particular,* $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X) = \mathrm{Hom}_{\mathbb{A}}\,(\mathbb{A}, \mathrm{X_e})$.

Proof. Since $X_e \subseteq X$, clearly $\mathrm{Hom}_{\mathbb{A}}\,(E, X_e) \subseteq \mathrm{Hom}_{\mathbb{A}}(E, X)$. Now let $T \in \mathrm{Hom}_{\mathbb{A}}(E, X)$. Then, for any $x \in E$, since, $X \to x$,

$$\lim T(x) = \lim T\,(e_\lambda x) = T(x) \cdot \lambda.$$

Therefore $T(x) \in X_e$, i.e. $T \in \mathrm{Hom}_{\mathbb{A}}\,(E, X_e)$. □

Further, we obtain the following results that generalize some results of [11, 38, 57, 78] to modules of continuous homomorphisms.

Proposition 5.3.1. *Let* X *be a topological left* A*−module. Then:*

(i) If X *is an* $\mathbb{F}$*-space and* $\mathbb{A}$ *is strongly factorable, then both* $(\mathrm{Hom}_{\mathbb{A}}(X), \mu)$ *and* $(\mathrm{Hom}_{\mathbb{A}}(\ \mathbb{A}, \mathrm{X}), \mathrm{p})$ *are complete.*

(ii) If X *is complete and* $\mathbb{A}$ *is ultrabarrelled having a bounded approximate identity, then both* $(\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X), \mu)$ *and* $(\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_{\mathbb{A}}), \mathrm{p})$ *are complete.*

Proof. (i) Let $\{T_\alpha : \alpha \in J\}$ be a u-Cauchy net in $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$. Since $p \leqslant u, \{T_\alpha : \alpha \in J\}$ is a p-Cauchy net in $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$; in particular, for each $a \in A, \{T_\alpha(a)\}$ is a Cauchy net in $\mathbb{A}$. Consequently, by completeness of X, the mapping $T : \mathbb{A} \to X$, given by $T(a) = \lim_\alpha T_\alpha(a)$ such that $a \in \mathbb{A}$), is well defined. Further, for any $a, b \in \mathbb{A}$,

$$T(ab) = \lim T_\alpha(ab) = a \lim{}_\alpha T(b) = \mathrm{a} \cdot \mathrm{T(b)}.$$

Since X is metrizable and $\mathbb{A}$ strongly factorable, $T \in \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$.
We now show that $T_\alpha \xrightarrow{\mu} T$. Let D be a bounded subset of $\mathbb{A}$ and take closed $G \in W_x$. There exists an index α_0 such that

$$\mathrm{T}_\alpha(\mathrm{a}) - \mathrm{T}_x(\mathrm{a}) \in \mathrm{G} \quad \text{for all } a \in D \text{ and } \alpha, \gamma \geqslant \alpha_o.$$

Since G is closed, fixing $\alpha \geqslant \alpha_o$ and taking $\lim_\gamma$, we have

$$T_\alpha(a) - T_\gamma(a) \in G \quad \text{for all } a \in D.$$

Thus $(\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X), \mu)$ is complete. By a similar argument, $(\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X), p)$ is also complete. Hence $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_e)$ is also p-complete. □

Definition 5.3.2. *For any* $x \in X$*, define* $Rx : \mathbb{A} \to X$ *by* $Rx(a) = a.x, a \in \mathbb{A}$*. Clearly,* Rx *is linear and continuous (by separate continuity of module multiplication); further, for* *any* $a, b \in \mathbb{A}$,

$$R_x(ab) = (ab) \cdot x = a \cdot (b \cdot x) = a \cdot R_x(b) \in X\}$$

is a left $\mathbb{A}-$ *submodule of* $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$.

We now give a result on the density of the functors.

Proposition 5.3.2. *Let* X *be a topological left* $\mathbb{A}$*-module with* $\mathbb{A}$ *having a two-sided approximate identity* $\{\mathrm{e}_\lambda : \lambda \in \mathrm{I}\}$. *Then* $\mu\left(\mathrm{X_e}\right)$ *is* p*-dense in* $\mathrm{Hom}_{\mathbb{A}}\left(\mathbb{A}, \mathrm{X_a}\right)$*; in particular,* $\mu(\mathrm{X})$ *is* p*-dense in* $\mathrm{Hom}_{\mathbb{A}}\left(\mathbb{A}, \mathrm{X}_{\mathbb{A}}\right)$.

Proof. Let $T \in \mathrm{Hom}_{\mathbb{A}}\left(\mathbb{A}, X_{\mathbb{A}}\right)$. For each $\lambda \in I$, define $x_\lambda = T\left(e_\lambda\right)$. Then

$$\lim e_\gamma \cdot x_\lambda = \lim e_\gamma \cdot T\left(e_\lambda\right) = \lim T\left(e_\gamma e_\lambda\right) = T\left(e_\lambda\right) = x_\lambda : \gamma \leq \lambda$$

and so $x_\lambda \in X_e$.
Now, for any $a \in \mathbb{A}, a \cdot x_\lambda = a \cdot T\left(e_\lambda\right) = T\left(a_\lambda\right) \longrightarrow a$; hence $\mu\left(x_\lambda\right)(a) = \mu(Te_\lambda)(a)$

$$= a \cdot x_\lambda = a \cdot T(e) = T(aex) \to T(a).$$

Therefore $\mu\left(x_{\mathbb{A}}\right) \longrightarrow$ T. That is, $\mu\left(X_e\right)$ is p-dense in $\mathrm{Hom}_{\mathbb{A}}\left(\mathbb{A}, X_e\right)$.

□

Remark 5.3.1. *Note that* $\mu(X)$ *need not be u-closed in* $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$ *even if* $X = \mathbb{A}$ *is a metrizable locally* C^**-algebra (see [56]) or a Banach algebra [78]. However, if* $X = A$ *is a* B^**-algebra or, more generally, an F-algebra whose topology is generated by a submultiplicative F-norm q (cf. [81]) such that* $q\left(e_\lambda\right) = 1$ *for all* $\lambda \in I$*, then* $\mu : \mathbb{A} \to (\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A}), \mu)$ *is an isometry, as follows:*
Let $x \in X = \mathbb{A}$. *Then*

$$a_{a \neq 0} \quad R_x\|_q = \sup \frac{q\left(R_x(a)\right)}{q(a)} = \sup \frac{q(ax)}{q(a)} \leqslant \sup \frac{q(a)q(x)}{q(a)} = q(x).$$

On the other hand,

$$R_x\|_q = \sup_a \frac{q(ax)}{q(a)} \geqslant \frac{q\,(xe_\lambda)}{q\,(e_\lambda)} \geqslant q\,(xe_\lambda) \qquad \textit{for all } \lambda \in \mathrm{I}.$$

So

$$R_x\|_q \geqslant \lim{}_\lambda q\,(e_\lambda x) = q\,(\lim{}_\lambda e_\lambda x) = q(x).$$

Hence $R_x\|_q = \hat{q}(x)$. Therefore μ is an isometry; hence $\mathbb{A}$ is u-closed in $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A})$.

Definition 5.3.3. *Let (X, τ) be a topological left $\mathbb{A}-$module, where $\mathbb{A}$ is a topological algebra, and let W_x be a base of neighbourhoods of 0 in X. For any bounded set $D \subseteq \mathbb{A}$ and $G \in W_x$, we set*

$$N(D, G) = \{x \in X : D \cdot x \subseteq G\}.$$

The uniform topology $\tau' = \tau'_{\mathbb{A}}$ (respectively general strict topology $\beta = \beta_{\mathbb{A}}$) on X is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form $N(D, G)$, where D is a bounded (respectively finite) subset of $\mathbb{A}$ and $G \in W_x$.

It is shown in [38] that;

(i.) $\beta \leqslant \tau'$

(ii.) if the module multiplication is b($\mathbb{A}$)-hypocontinuous (in particular, X is ultrabarrelled), then $\tau' \leqslant \tau$

(ii.) if $\mathbb{A}$ has a two-sided approximate identity (and (X, τ) is Hausdorff), then β and τ' are Hausdorff.

Note that, for any bounded set $\mathrm{D} \subseteq \mathbb{A}$ and $\mathrm{G} \in \mathrm{W_x}$, we have

$$M(D, G) \cap \mu(X) = \{T \in \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X) : T(D) \subseteq G\} \cap \mu(X) = \{R_x : x \in X, R_x(D) \subseteq G\} = \mu(N(D, G)$$

hence τ is the topology of bounded convergence of $\mathrm{Hom}_{\mathbb{A}}\,(\mathbb{A}, X_e)$ induced on X under the algebraic embedding μ.

Considering $Y = \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$ as a left $\mathbb{A}$-module, we can also define the strict topology $\beta = \beta_{\mathbb{A}}$ on $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X)$ as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(D, G) = \{y \in Y : D * y \subseteq G\}$$

where D is a finite subset of $\mathbb{A}$ and $G \in W_Y$.
We mention that if $X = \mathbb{A} = C_0(S)$ with S a locally compact Hausdorff space, then $\mathbb{A}$ is a commutative Banach algebra having a bounded approximate identity and $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A}) = M(A) = C_b(S)$ [78].

Next, we investigate the completeness of both (X, β) and $(\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X), \beta)$.

Proposition 5.3.3. *Let* X *be a topological left* $\mathbb{A}$*-module with* $\mathbb{A}$ *having a two-sided approximate identity* $\{\mathrm{e}_\lambda : \lambda \in \mathrm{I}\}$*. If* (X, β) *is complete, the map* $\mu : X \to \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathrm{X})$ *defined by* $\mu(\mathrm{y}) = \mathrm{R}_v \in \mathrm{X}$*, is onto.*

Proof. μ(X) is p-dense in $\mathrm{Hom}_{\mathbb{A}}\,(\mathbb{A}, X_{\mathbb{A}})$.
We now show that $\mu(X)$ is p-closed in $\mathrm{Hom}_{\mathbb{A}}\,(\mathbb{A}, X_e)$. Let p $\mathrm{T} \in \mathrm{p} - \mathrm{cl}\,\mu(\mathrm{X})$. There exists a net $\{\mathrm{x}_\alpha : \alpha \in J\} \subseteq \mathrm{X}$ such that $\mathrm{R}_{\mathrm{x}\alpha} \longrightarrow \mathrm{T}$. Then $\{\mathrm{x}_\alpha : \alpha \in J\}$ is β-Cauchy in X. [Let D be a finite subset of A and $G \in W_x$. Choose a balanced $H \in W_x$ with $H + H \subseteq G$. Since $R_{x\alpha} \longrightarrow T$, there exists $\alpha_0 \in I$ such that for all $\alpha \geqslant \alpha_o$,

$$R_x\alpha - T \in N(D, H) \quad \text{ or } \quad R_{xa}(a) - T(a) \in H \quad \text{for all a } \in D.$$

Then, for any $a \in D$ and $\alpha, \gamma \geqslant \alpha_o$,

$$a \cdot x_\alpha - a \cdot x_\gamma = [R_{x_\alpha}(a) - T(a)] + \left[T(a) - R_{x_\gamma}(a)\right] \in H + H \subseteq G.$$

Since (X, β) is complete, $x_\alpha \longrightarrow x_\alpha x_\gamma \in X$. Hence $R_{x\alpha} \longrightarrow R_{xe}$. By uniqueness of limit in Hausdorff spaces, $T = R_{xq} \in \mu(X)$. Thus $\mu(X) = \mathrm{Hom}_{\mathbb{A}}\,(\mathbb{A}, X_e)$.

□

Theorem 5.3.1. *Let* X *be a left* $\mathbb{A}-$ *module with* X *complete and* $\mathbb{A}$ *ultrabarrelled having a bounded approximate identity. If the map* $\mu : X \longrightarrow \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}_{\mathbb{A}}, X_e)$ *is onto, then* (X, β) *is complete.*

Proof. Let $\{x_\alpha\}$ be a β-Cauchy net in X. Therefore $\{R_{x\alpha}\}$ is a p-Cauchy net in $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_{\mathbb{A}})$ is p-complete, and so $R_{x\alpha} \longrightarrow T$ in $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_{\mathbb{A}})$. Since μ is onto, there exists $x_0 \in X$ such that $p\,T = R_{xe}$. Therefore $R_{x\alpha} \longrightarrow R_{xe}$.
Now, let D be a finite subset of A and $G \in W_x$. Since $R_{x\alpha} \longrightarrow R_{xe}$, there exists $\alpha_0 \in I$ such that for all $\alpha \geqslant \alpha_o$

$$R_{x\alpha} - R_{xq} \in M(D, G) \quad \text{or} \quad x_\alpha - x_0 \in N(D, G).$$

Hence $x_\alpha \longrightarrow x_0$, and so (X, β) is complete. □

Theorem 5.3.2. *Let* (X, τ) *be a topological left* $\mathbb{A}$*-module with* $\mathrm{b}(\mathbb{A})$*-hypocontinuous module multiplication. Suppose* (X, T) *is complete and* $\mathbb{A}$ *has a bounded approximate identity* $\{\mathrm{e}_\lambda : \lambda \in \mathrm{I}\}$*. Then* $([\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathrm{X_e})]_{\mathrm{e}}\,\tau)$ *is topologically isomorphic to* $(\mathrm{X_e}, \tau)$*.*

Proof. The proof follows from [1].

□

Theorem 5.3.3. *Let* (X, τ) *be a topological left* $\mathbb{A}$*-module with* $\mathrm{b}(\mathbb{A})$*-hypocontinuous module multiplication. Suppose* (X, τ) *is complete ultrabarrelled and* $\mathbb{A}$ *is ultrabarrelled and ultrabornological as a TVS and has a uniformly bounded two-sided approximate identity* $\{\mathrm{e}_\lambda : \lambda \in \mathrm{I}\}$*. Then* $(\mathrm{Hom}_{\mathbb{A}}(\ \mathbb{A}, \mathrm{X}_0), \beta)$ *is complete.*

Proof. Let $Y = \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_e)$. Since X is ultrabarralled and $\mathbb{A}$ has a bounded left approximate identity, X_e is $\tau-$ closed and hence $\tau-$complete. Then, $L(\mathbb{A}, X_e)$ is $\mu-$complete. Since $\mathbb{A}$ is ultrabornological, $L(\mathbb{A}, X_e) = Y_e$ given by $\mu(x) = R_x x \in X_e$, is a topological isomorphism. We observe that given $T \in Y_e$, there exists $z \in X_e, z = \lim_\lambda T(e_\lambda)$ and $T = T_z$.

Define a map $\sigma : Y \to \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, Y_e)$ by $\sigma(T)(a) = a * T_a \in \mathbb{A}, T \in Y$. We need to show that σ is onto.

Let $S \in H_0$ Hom $_{\mathbb{A}}(\mathbb{A}, Y_e)$. We claim that $R \in \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_e)$. [Since $S(a) \in Y_e, \lim_\lambda$ and $S(a)(e_\lambda)$ exists in X_e. Clearly, R is linear.

Further:

$$R(ab) = \lim{}_\lambda S(ab)(e_\lambda) = \lim{}_\lambda [a \cdot S(b)](e_\lambda) = \lim{}_\lambda S(b)(e_\lambda a) = S(b)(a),$$

$$a.R(b) = a_\lambda \cdot \lim{}_\lambda S(b)(e_\lambda) = \lim{}_\lambda \mathrm{a} S(b)(e_\lambda) = \mathrm{limS}(b)(\mathrm{ae}_\lambda) = S(b)(a)$$

hence R is a left $\mathbb{A}$-homomorphism of $\mathbb{A}$ into X_e.

We next show that R is continuous. First, for each $\lambda \in I$, define $S_{Sd} : \mathbb{A} \to X_e$ by

$$S_{\mathbb{A}}(a) = S(a)(e_\lambda), \quad a \in \mathbb{A}.$$

Clearly, each S_λ is a linear map; further, by continuity of S and separate continuity of module multiplication, S_λ is also continuous. Now $R(a) = \lim_\lambda S(a)(e_\lambda) = \lim_\lambda S_\lambda(a)$ and $\mathbb{A}$ is ultrabarrelled, it follows from uniform boundedness principle that R is continuous. Therefore $R \in \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, X_e)$.] We now show that $\sigma(R) = S$. Let $a \in \mathbb{A}$. Then, for any $b \in \mathbb{A}$,

$$\sigma(R)(a)(b) = (a * R)(b) = R(ba) = b.lim_\lambda S(a)(e_\lambda)$$

$$= \lim.\mathrm{b}\, S(a)(e_\lambda) = \mathrm{limS}(a)(bex) = S(a)(b).$$

Thus $\sigma : Y \to Hom_{\mathbb{A}}(\mathbb{A}, Ye)$ is onto. Consequently, (Y, β) is complete. $\square$

CHAPTER SIX

CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

In this chapter, we provide the conclusion and a raft of recommendations from our study.

6.2 Conclusion

Due to the immense applications in Spectral Theory, Geometry of Banach spaces, Theory of eigenvalue distributions among others, the Theory of operator ideals and modules occupies a special importance in functional analysis. Therefore, the main objective of this study was to characterize the algebra of ideals and modules in operator spaces. This has been achieved via four main specific objectives. First, in chapter three, we have given an account on the ideals in Banach spaces where the various characterizations of $L, M, u, h-$ideals and their variants have been determined up to a classification. Here, classes of: compact operators, bounded linear operators, finite ranks operators in relation to their ideal properties have been exhibited. The general properties of the M-ideals have been studied; key and relevant results presented. In particular, the interplay among the ideals mentioned as well as their extensions have been established. The result in Proposition 3.2.1 demonstrates the existence of classes of closed operator ideals. Theorem 3.2.1 gives the boundedness in view of Radon-Nickodym properties. Additionally, the findings of Theorem 3.3.1 give the characteristics of ideals through the Hahn-Banach extension opertors. Proposition 3.4.1 characterizes L, $M-$ ideals using the standard projections. Moreover, Proposition 3.5.1 provides necessary and sufficient conditions for the existence of $u, h-$ideals and their variants in generalized Banach spaces. In section 3.10.1, the work focussed on $u-$ideals in their biduals with a keen analogy on separability and norm attainability. The result of Theorem 3.10.1 gives the details.

In chapter four, spaces of ideal operators are studied. By the use of approximation properties, we characterized operator ideals. The u-ideals and their local and hereditary properties have been studied. Using the Hahn-Banach extension, a study on strict u-ideals is done and shown that if you have a Banach space which is a strict u-ideal and has a separable subspace, then the separable subspace is a strict u-ideal and it is separably determined. Proposition 4.4.1 gives the metric approximation property in classes of tensorially well defined Hahn-Banach extensions and equivalent renorming. Theorem 4.6.1 gives renorming of $u-$ideals as an operator space. Theorem 4.9.3 looks at integral property, $\lambda-$boundedness and $\lambda-$boundedness approximation property and establishes the connectedness of the three properties.

Finally, chapter five looked at the topological interplay between ideals and modules since idealization allows borrowing of ideal properties and using them to study modules. First, we give preliminary studies concerning Banach Rings and Banach Modules with particular focus on Archimedean and non-Archimedean operator spaces (Banach Spaces). We give two accounts. First, the relationship between categorical products and co-products of kernels from one module to other. The results of Theorem 5.2.2 shows that any two projective modules in an arbitrary Banach algebra have a flat tensor product in the Algebra. Similarly, Theorem 5.2.3 shows that given an inductive system of modules in a bounded Banach algebra, all canonical morphisms from the module to the collections of its isometric immersions is an isomorphism. The chapter extends to the study of topological vector spaces using the standard Fretchet spaces as a baseline. Certain characteristics of Fretchet modules including strong factorization properties over the Functors have been given. The continuity and hypo-continuity of the multiplication have been shown. In particular, Theorems 5.3.2 and 5.3.3 are among the main results.

6.3 Recommendations

In view of the above conclusion, we recommend that future studies can consider the following areas:

(i.) A characterization of semi-simple modules in Banach Algebras in relation to one sided ideal structures.

(ii.) On new classes of operator ideals in generalized spaces determined by $M, u, h-$ projections.

(ii.) On the Category of Ideals and Modules on $\mathbb{C}_0$ and $\mathbb{C}^*$ spaces.

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