CERTAIN GEOMETRIC ASPECTS OF A CLASS OF ALMOST CONTACT STRUCTURES ON A SMOOTH METRIC MANIFOLD

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DECLARATION

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ABSTRACT

The classification of Smooth Geometric Manifolds still remains an open problem. The concept of almost contact Riemannian manifolds provides neat descriptions and distinctions between classes of odd and even dimensional manifolds and their geometries. Among the classes that have been extensively studied in the past are the Hermitian, Symplectic, Khalerian, Complex, Contact and Almost Contact manifolds which have applications in M-Theory and supergravity among other areas. The differential geometry of contact and almost contact manifolds and hence their applications can be studied via certain invariant components: the structure tensors, connections, the metrics and the maps. The study of almost contacts $1, 2, 3$-manifolds has been explored before to an extent. However, little known is the existence and the geometry of an almost contact 4-structure. In this thesis, we have constructed a class of almost contact structures which is related to almost contact 3-structure carried on a smooth Riemannian metric manifold (M, g_M) of dimension $(5n + 4)$: gcd $(2, n) = 1$. Starting with the almost contact metric manifold (N^{4n+3}, g_N) endowed with structure tensors (ϕ_i, ξ_j, η_k) of types (1,1),(1,0),(0,1) respectively, for all $i, j, k = 1, 2, 3$, we have showed that there exists an almost contact structure (ϕ_4, ξ_4, η_4) on $(N^{4n+3} \otimes \mathbb{R}^d) \approx M^{5n+4}$; $gcd(4, d) = 1$ and $d|(2n + 1)$ constructed as a linear combinations of the first three structures on (N^{4n+3}, g_N). We have studied the geometric properties of the tensors of the constructed almost contact structure, the properties of the characteristic vector fields of the two manifolds M^{5n+4} and N^{4n+3} and the relationship between them via an α-rotated submersion $\Pi : (N^{4n+3} \otimes \mathbb{R}^d) \hookrightarrow (N^{4n+3})$ and the metrics g_M respective g_N. This provides new forms of Gauss-Weingartens' equations, Gauss-Codazzi equations and the Ricci equations incorporating the submersion other than the First and Second Fundamental coefficients only. We have observed that the almost contact structure (ϕ_4, ξ_4, η_4) is constructible if and only if it is carried on the hidden compartment of the manifold $M^{5n+4} \cong (N^{4n+3} \otimes \mathbb{R}^d)$ which is related to the manifold N^{4n+3}. The results of this study establish a strong basis upon which the study of almost contact structures can be extended to more than 4-structures. Moreover, the fact that the vector field $\{\xi_i : \ i = 1, \cdots, 4\}$ obtained is killing gives rise to integral geodesic curves which allow for smooth interpolation between two high-dimensional points with application in computer vision where smooth animations can be constructed by travelling along the geodesics between two images. These manifolds can thus be applied in the exploration of M-theory and supergravity.

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DEDICATION

God Almighty: my creator, my strong pillar, my source of inspiration, Wisdom, Knowledge and understanding. He has been the source of my strength throughout this program and on His wings only have I soured.

My husband: Barnabas Rotich who has encouraged me all the way and whose encouragement has made sure that I give it all it takes to finish that which I have started. God bless you.

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My brothers and sisters: Hoping that this thesis I have proven to you that there is no mountain higher as long as God is on our side. Hoping that you will walk again and be able to full fill your dreams.

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INDEX OF NOTATIONS

ϕ - A field of Endomorphisms .

ξ - Vector field .

η - One form .

$T(M)$ - Tangent bundle.

$\otimes$ - Tensor product.

$\wedge$ - Wedge product.

∇ - Levi-civita connection.

TM^* - Cotangent bundle.

T_pM - Tangent bundle at some point $p \in M$.

$\oplus$ - Direct sum.

M - Manifold.

M^n - Manifold of dimension n.

$M^n \times \mathbb{R}$ - The product of a manifold of M^n with real line R.

$[\phi, \phi]$ - Ninjehuis tensor of ϕ.

C^∞ - Infinite complex field .

$\circ$ - Composition of maps.

$U(n)$ - Unitary group of n complex variables.

$d\eta$ - First differential of the form.

$\sum$ - Summation.

Q - Almost quaternionic structure.

J - Almost complex structure.

g - Metric.

I - Unit tensor.

ω - 2-form.

$\perp$ - Orthogonal complement.

Γ - The smooth section of the given map.

$GL(n, \mathbb{R})$ - General linear group.

CHAPTER ONE

INTRODUCTION

1.1 Background of study

Contact geometry was revealed in 1896 by a research carried out by Sophus Lie on partial differential equations and since then, it has been used as an important tool to study odd-dimensional manifolds as an analogy of symplectic geometry in even-dimensional manifold. The subject has intrinsically shown to be underlying many physical phenomena and closely associated to many other mathematical structures. For example, the Gibbs' work on thermodynamics, Hygens' work on geometric optics and in Hamiltonian Dynamics were both revolving around it. Moreover, contact and almost contact structures have relations with fluid mechanics, Riemannian geometry, Low Dimensional Topology and provide interesting classes of subelliptic operators, Geiges [15]. Due to the significant applications of contact and almost contact geometry, research in this line has been active.

Borman et'al [6] have shown that any closed odd-dimensional manifold with almost contact structures usually admits at least a contact structure. They proved the existence of overtwisted contact structures and classified them for all dimensions. Their classification was however not extended to almost contact manifolds. Sergey [31] Provided a slight extension of the work in [6] by considering an almost contact structures characterized by N-prolonged connection. It was noted that the metric structures studied were normal so that $N_\phi \oplus 2(d\eta \circ \phi) \otimes \xi = 0$. These structures later turned out to be Sasakian. Eliashberg [12] introduced a dichotomy of d-dimensional contact manifold: $d = 3$ into tight and overtwisted manifolds and established a parametric h-principle for overtwisted ones. He found that any almost contact homotopy class on a closed 3- dimensional manifold contains a unique upto isotopy, overtwisted contact structures.

Great achievement in the problem of classification of contact structures on a closed manifolds was achieved in the 5-dimensional case by Etnyre [13]. The research established the existence of contact structures on any 5- dimensional manifold, but in any homotopic class of almost contact structures for manifolds of dimensions > 5 the results are scarce. The study of almost contact manifolds was first introduced by Gray [16] in 1959, by defining an odd-dimensional manifolds whose structure group of tangent bundle can be reduced to $U(n) \times 1$. Some general properties of contact structures including the non-vanishing property of the volume form were established. Later, in 1960, Sasaki [29] introduced an equivalent definition of almost contact manifolds. In his research Sasaki found out some results that he took to Hatekayama [19] who proved cases when the structure of the group of any differentiable manifold M^{2n+1} reduceds $U(n) \times 1$, so the M^{2n+1} is considered a manifold with almost contact structure.

Geiges [15] studied contact structures on $(n-1)$ connected $(2n+1)$-dimensional manifolds and showed that contact structures exist on simply connected 5-dimensional manifolds by applying results on contact surgery that was later extended by Borman et'al [6]. However, little has been done about the contact structures in high-dimensional manifolds. Differentiable manifolds with contact and almost contact structures were classically classified rather from a topological point of view (see [5, 16]). The study of the geometry of tangent bundle was investigated by Sasaki [29] . Using the Riemannian metric on a manifold M, Sasaki defined a Riemannian metric g^1 on the tangent space TM of the manifold M. This construction was grounded on the natural splitting which takes place due to the existence of Levi-civita connections of the tangent bundle TM of the manifold M into the direct sum of vertical and horizontal distribution, the fibres of these distributions are isomorphic to the fibres of distribution TM. According to Sergey [31] the odd analogy of the tangent bundle is a distribution of the almost contact structure (ϕ, ξ, η), similarly to the bundle TM, the bundle due to a connection over the distribution compartmentalizes into the direct sum of the vertical and horizontal distribution.

Adara [2] has studied almost k-contact structure, pointing out an isoparametric function which can be associated in this framework, by generalizing a similar construction initiated by Mihai and Rosca [27]. From Adaras' constructions, an almost k-contact manifold is found to be $(n+k+nk)$-dimensional manifold M with k-almost contact structures $(\phi_1, \xi_1, \eta_1), ..., (\phi_k, \xi_k, \eta_k)$ such that,

$$\phi_i \circ \phi_j = -\delta_{ij} I_{\Gamma TM} + \eta_i \otimes \xi_j + \sum_{l=1}^{k} \epsilon_{ij} l \phi l,$$

and $\eta_i(\xi_j) = \delta_{ij}$. Accordingly, given the almost contact 3-structure (ϕ_i, ξ_i, η_i), defined on $M^{2n+1} \times \mathbb{R}$ by Adara [1], there are three almost complex structures J_i such that $i = 1, 2, 3$ associated to each of the almost contact structures. It can be verified that $J_k = J_i J_j = -J_j J_i$ an anticommutativity condition of the structures. Therefore, $M^{2n+1} \times \mathbb{R}$ has an almost quarternionic structure and hence its dimension is a multiple of 4. Thus the dimension of an almost contact 3-structure is of the form $4n + 3$ Blair [7]. Tachibana and Yu [33] had initially used this idea to show that there cannot exist a fourth almost contact structure (ϕ_4, ξ_4, η_4) with $\eta_i(\xi_4) = \eta_4(\xi_i) = 0$ for all $i = 1, 2, 3$ and satisfying the anti-commutativity conditions with the first three structures: $(\phi_1, \xi_1, \eta_1), (\phi_2, \xi_2, \eta_2), (\phi_3, \xi_3, \eta_3)$ and the associated complex structures. To see this, Blair [7]demonstrated that if J_4 is the almost complex structure on $M^{2n+1} \times \mathbb{R}$ constructed using (ϕ_4, ξ_4, η_4), then pairing J_4 with each of J_1, J_2, J_3 yields $J_4 J_i = -J_i J_4$ such that $i = 1, 2, 3$. This contradicts $J_3 J_4 = J_1 J_2 J_4 = -J_1 J_4 J_2 = J_4 J_1 J_2 = J_4 J_3$ which is the conventional condition. In this thesis, we show that if an almost contact 3-structure is given, then there exists a structure (ϕ_4, ξ_4, η_4) that depends on the first three structures. The constructibility of this fourth structure disapproves Tachibana and Yu's conjecture. We have further explored the geometry of the manifold carrying (ϕ_4, ξ_4, η_4).

1.2 Basic Definitions

The definitions given below are standard and can be obtained from the references. They shall be used frequently in the document.

Definition 1.2.1. ***A manifold M*** *is a topological space with a maximal atlas or maximal smooth structure.*

Definition 1.2.2. ***Atlas of a Manifold*** *is a collection of charts whose domain cover the manifold.*

Definition 1.2.3. ***Fibre bundle*** *is a manifold that looks locally like a product of two manifolds, but is not necessarily a product globally.*

Definition 1.2.4. ***A smooth manifold*** (M, U) *is a topological manifold* M *equipped with a smooth structure* U.

Definition 1.2.5. ***Riemannian manifold*** *is a manifold on which one has defined a specific symmetric and positive definite(or non-singular) 2-covariant tensor field, known as metric tensor.*

Definition 1.2.6. ***Tangent space*** *is the set of all derivations of a smooth manifold at a point* p.

Definition 1.2.7. ***Tangent bundle*** *is the union of all tangent spaces to* M *denoted* TM.

Definition 1.2.8. ***Geodesic Curve*** *is the shortest curve connecting two points on a surface.*

Definition 1.2.9. ***Isometry*** *is a mapping of a portion of a manifold* M *to a portion of a manifold* N*, if the length of any curve on* N *is the same as length of its pre-image on* M.

Definition 1.2.10. ***Killing vector*** *A vector field* X *is called a killing vector field if the 1-parameter group of infinitesimal transformations generated by* X *is a group of isometries. Equivalently,* $L_{Xg} = 0$ *Blair [7].*

Definition 1.2.11. ***Diffeomorphism*** *A smooth map* f *between* M *and* N *such that* $f : M \to N$ *is called a diffeomorphism if* f *is bijective and* f *inverse is also smooth.*

Definition 1.2.12. ***Kernel*** *The kernel of* $f : M \to N$ *denoted* $Ker(f)$ *measures the degree to which the homomorphism* f *fails to be injective.*

1.3 Statement of the problem

The discovery of almost contact Riemannian manifolds has contributed immensely to advancement of both Geometry and Algebraic Geometry. Due to the unique properties of the tensors associated with any almost contact structure, their geometry has resulted to more advanced applications of the ambient manifolds in Mathematics, Computer Science and Engineering. For example, the vector bundle TM of an almost contact manifold M contains nonparallel vector fields ξ; which are killing vectors giving rise to geodesic curves on the vertical distribution of the manifold hence, if applied in computer analysis, allows for smooth interpolation between 2-dimensional points feasible in getting smooth animations. On the other hand, the first and second fundamental coefficients M, N, L are useful for determining the nature of curves based on their curvatures k, usually provided by Codazzi-Mainardi equations and applicable in the classification of curves. However, the manifolds considered in such cases are usually restricted to Euclidean space. For a general metric manifold, the classification is sufficient when submersions are considered between the ambient manifold and submanifolds. This has not been done before for an odd dimensional manifold carrying more than 2 almost contact structures. The study of almost contact 1,2,3-manifolds has been explored before by a number of Geometers to an extent (see for example [5],[20],[27],[28]). However, little known is whether there exists an almost contact 4-structure on any odd dimensional manifold M. In fact, given 2 almost contact structures, Kuo [20] proposed a necessary condition for a third almost contact structure to exist on N^{4n+3} but did not provide the sufficiency hence the validity of the structure by proving the structure tensor properties. Moreover Tachibana and Yu [33] conjectured the non-existence of a fourth almost contact structure on any odd dimensional manifold and satisfying the anti-commutativity condition. This research provides a proof of the sufficiency for existence of a third almost contact structure and further constructs a fourth almost contact structure (ϕ_4, ξ_4, η_4) on the manifold $M^{5n+4} \cong (N^{4n+3} \otimes \mathbb{R}^d)$; $d|(2n+1)$ and $gcd(2, n) = 1$. Finally, the study explores the geometry of the submersion between the manifold carrying 4 structures and the one carrying 3 structures giving rise to new forms of Gauss, Weingarten, Codazzi and Ricci equations.

1.4 Objectives of study

1.4.1 Main Objective

To study certain geometric aspects of a class of almost contact structures on a smooth metric manifold.

1.4.2 Specific Objectives

(i) To construct a fourth almost contact structure (ϕ_4, ξ_4, η_4) on $(N^{4n+3} \otimes \mathbb{R}^d) \cong M^{5n+4}$ from (ϕ_i, ξ_i, η_i) for $i = 1, 2, 3$ on N^{4n+3}, where the gcd; $(4, d) = 1$, $(2, n) = 1$ and $d|(2n + 1)$.

(ii) To determine the geometric properties of the tensors, the Reeb vector fields and the metrics of the constructed almost contact structure (ϕ_4, ξ_4, η_4).

(iii) To determine new forms of Gauss-Weingarten, Gauss-Codazzi and Ricci equations via a submersion between the two manifolds: M^{5n+4} and N^{4n+3}.

1.5 Methods of study

The methods adopted for this study involved the following stages:

(i) We performed a cursory combinatorial analysis on the three almost contact structures (ϕ_i, ξ_i, η_i) $i = 1, 2, 3$ and determined a finite number of such combinations giving rise to a fourth almost contact structure. This method involved inductive algorithms.

(ii) We employed the standard structural criterion on almost contact structure as postulated in Blairs' Theorem [7] on Contact and almost Contact geometry.

(iii) We used the fundamental form procedures involving α-rotated submersions $F : M^{5n+4} \hookrightarrow N^{4n+3} \otimes \mathbb{R}$ to investigate and characterize the structural relationships between the manifolds.

1.6 Significance of the study

These classes of manifolds have application in computer, graphics and augment reality given the need to associate pictures that is texture to coordinates for example CT scans. Riemannian metric on a manifold allows distances and angles to be measured. In view of Riemannian metric, if the data space is geodesic, the space would allow for smooth interpolation between two high-dimensional points: this may have applications in computer vision, where smooth animation between images can be constructed by travelling along geodesics between the two images. This may also redefine a potential to revolutionize machine learning technique such as dimension reduction and clustering by providing a more accurate measure of distance in data spaces than Euclidean distance prevalent. The results of this study also establish a strong basis upon which the study of almost contact structure can be extended to more than 4-structures.

CHAPTER TWO

LITERATURE REVIEW

2.1 Riemannian Geometry

Adara [1] explored the geometry of lightlike submanifolds in metallic semi-Riemannian manifold M and proved that the metric induced on M of Riemannian type is always a Riemannian one, however in semi-Riemannian manifolds the metric induced by the semi-Riemannian metric on the ambient manifold is not always non-degenerate. This result provided an important class of submanifolds referred to as lightlike submanifolds, due to the degeneracy of the induced metric on lightlike submanifolds, the tools which are used to investigate the geometry of submanifolds in Riemannian case are not applicable in semi-Riemannian case and so the classical theory fails while defining any induced object on a lightlike submanifolds. Hakan [18] studied Lagragian submanifolds of Normal almost contact manifolds from Sasakian and Kenmotsu manifolds onto Riemannian manifolds by showing that the horizontal distribution of a Lagragian submanifolds from a Sasakian manifold onto a Riemannian manifold admitting vertical Reeb vector field is integrable but the one admitting horizontal Reeb vector field is not, therefore the horizontal distribution of such submanifolds is integrable when the total manifold is Kenmotsu. Bayram and Mehmet [4] studied conformal semi-invariant submersion from Hermitian manifolds onto Riemannian manifolds by investigating the geometric foliations which come from the definition of a conformal submersion and showed that there are certain product structures on the total space of a conformal semi-invariant submersion. They finally checked the harmonicity properties of the submersions to find out the necessary and sufficient conditions for conformal semi-invariant submersion to be totally geodesic. The main method that was appropriate for this study was comparing the two manifolds and transfering certain structures from one manifold to another manifold by defining appropriate maps between them. Their studies however did not consider providing the Gauss, Weingarten, Codazzi and Ricci equations having the invariants of the submanifolds studied.

Tshikuna [37] studied almost contact metric submersions and the relationship between the properties of total space, the base space and the fibres by showing that the superminimality of the fibres is a tool in the transference of the structure from the ground to the total space. Later in the study on superminimality fibres in an almost contact metric submersion, Tshikuna proved that the superminimality of fibres plays an important role in the integrability of the horizontal distribution for almost contact metric submersions. Mehmet [26] studied submanifolds of Riemannian product manifold and generalized the geometry of invariant submanifolds of a Riemannian product manifold to the geometry of semi-invariant submanifolds of a Riemannian product manifold and considered necessary conditions and sufficient conditions given for semi-invariant submanifolds to be D-geodesic ($D^{\perp}$ geodesic) and mixed geodesic submanifold. Senlin and Yilong [30] later updated Matsumoto's Theorem and proved that (M_1, g_1) and (M_2, g_2) are pseudo-umblical submanifolds of (M_1', g_1') and (M_2', g_2') respectively, if (M, g) is an invariant pseudo-umblical submanifold of $(M_1' \times M_2', g_1' \times g_2')$. They also demonstrated that M is isometric to the production of its two totally geodesic submanifolds (M_1, g_1) and (M_2, g_2) which are submanifolds of (M_1', g_1') and (M_2', g_2') respectively.

The index of a metric plays significant roles in differential geometry as it generates variety of vector fields such as space-like, time-like and light-like fields, with the help of these vector fields one can establish interesting properties on ε-Sasakian manifolds which was introduced by Duggal and Bejancu [10]. In order to resolve the difficulties that rise during studying lightlike submanifolds Duggal and Bejancu [10] introduced non-degenerate distribution called screen distribution to construct a lightlike transversal vector bundle which does not intersect to its lightlike tangent bundle. It is well known that a suitable choice of screen distribution gives rise to many substantial characterization in lightlike geometry[10]. Different kinds of geometric structures such as almost product, almost contact, almost paracontact among others allow to get rich results while studying on submanifolds. It was observed that, Riemannian manifolds with metallic structures are some of the most studied topics in differential geometry Duggal and Bejancu [10]. Matsumoto [23] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it Lorentzian almost paracontact manifolds. On the other hand, in a Lorentzian almost paracontact manifold given by Matsumoto [23], the semi-Riemannian metric has only index 1 and the structure vector field (ξ) is always timelike. Hence, association of a semi-Riemannian metric not necessarily Lorentzian with an almost paracontact structure called indefinite almost paracontact metric structure on ε-almost paracontact structure, where the structure vector field (ξ) is spacelike or timelike are $\varepsilon = 1$ or $\varepsilon = -1$ respectively. It can be noted that the geometry of a submanifold (M, g) of a Riemannian product manifold $(M_1' \times M_2', g_1' \times g_2')$ has been studied by many geometers. Particularly, Matsumoto [24] who proved that (M, g) is a locally Riemnnian product manifold of Riemannian manifolds (M_1, g_1) and (M_2, g_2) if (M, g) is an invariant submanifold of a Riemannian product $(M_1' \times M_2', g_1' \times g_2')$. The problem of classification of manifolds via their geometry is thus still open.

2.2 Almost Complex Manifolds

An even dimensional manifold M with a tensor field J of type $(1, 1)$, that is (M, J) is called an almost complex manifold if $J^2 = -1$ Blair [7]. The structure J is integrable if it can induce a complex Manifold. By the Newlander-Nirenberg Theorem (see Wan[39]), J is integrable if and only if the Nijenhuis tensor vanishes that is:
$N(J)(X, Y) = J[JX, Y] + J[X, JY] - [JX, JY] + [X, Y] = 0$ Blair [7]. In 4-dimension, one can construct many compact almost complex manifolds without any complex structure. However, up to now it has been difficult to find a single higher dimensional manifold with almost complex structures but no complex structure. Wan [39] studied the curvature and integrability condition of an almost complex structure where it was observed that one can construct an almost complex structure on 6-sphere by using quaternions but this almost complex structure is not integrable; it is still an outstanding problem to determine the complex structure on 6-sphere. Indeed 6-sphere is a touchstone to understand the complex

structures of higher dimensional manifold Wan [39].

Evidently, classes of complex and almost complex manifolds have elicited a great deal of research in the recent times. Bryant [8] in a follow up research concerning Chern's study of almost complex structures on the 6-sphere aimed at the idea of exploiting the special properties of its well known almost complex structure J invariant under the exceptional group G_2 revealed that it was not possible to determine whether a 6-sphere has an integrable almost complex structure J. The results however did prove a significant identity that resolves the question for an interesting class of almost complex structure J on 6-sphere Bryant [8]. This showed that the symmetric group of almost complex structure J preserves both a metric g and 2-form ω on 6-sphere. These intrinsic properties of (M, J); $M = S^6$ can be extended to high dimensional almost contact structures; a concept that began more than sixty years ago when Ehresmann introduced in differential geometry, the notion of almost complex structures on a differentiable manifold of even dimensions Mekri [25].

Le Brun [21] had proved that there is no complex structure on 6-sphere that is compatible with metric g while Chern's identity can be interpreted to mean that there is no complex structure on 6-sphere that is compatible with 2-form ω. It turns out that these two cases are quite different, the condition of compatibility with metric g is a system of 12 point-wise algebraic equations on an almost complex structure J and as Le Brun's analysis shows that integrability conditions for such an almost complex structure J forms an involutive system whose general local solution depends on three holomorphic functions of three complex variables. In contrast, the condition of compatibility with ω is a system of only 6 point wise algebraic conditions on an almost complex structure J, while Chern's identity shows that the integrability conditions for such almost complex structures do not form an involutive system. Indeed, his computation uncovers the non vanishing torsion that proves its non-involutivity.

A hyper complex structure on real vector space V is a triple (I, J, K) of complex structures on V satisfying the equation $IJ = K$. One important difference between complex and hyper complex geometry is the existence of special connection ∇ such that;

$\nabla I = \nabla J = \nabla K = 0$, Dominic [9]. Complex and hyper complex manifolds can be described clearly in terms of G-structures on manifolds. For instance, suppose P is the principal frame bundle of M that is the $GL(n, \mathbb{R})$ the bundle fibre over X in M is the group of isomorphisms $T_X M \cong \mathbb{R}^{4n}$, for a lie subgroup G of $GL(n, \mathbb{R})$, a G-structure Q on M is a principal sub-bundle of P with structure group G. The bundle Q admits a torsion-free connection if and only if there is a torsion-free linear connection ∇ on M with $\nabla I = 0$, where I is integrable Dominic [9]. Naturally there exists a structural relationship between the complex structure J and the almost contact structure (ϕ, ξ, η), the connection ∇ on M induces another connection ∇' on (ϕ, ξ, η) such that an extension remains uncovered in complex geometry.

2.3 Contact and Almost Contact Manifolds

Contact and almost contact manifolds are two classes of odd dimensional manifolds exhibiting closest structural relationships. The contact manifolds are those classes whose volume forms do not vanish while the almost contact are both contact and also satisfy three additional algebraic tensor properties on their structures. That is :
$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0$. They are fundamentally important as they have found celebrated applications in many areas of Mathematics, Science, Engineering and Computer Science. For instance, the interpretation of the principle of super-gravity has been made possible by using certain unique metric tensors embedded on almost contact structure. For this reason, research concerning the geometry of contact and almost contact manifolds has been continually advanced by a number of Geometers (see for instance [2, 3, 5, 11] e.t.c).

Todd [36] considered almost contact structures on G_2-manifolds and characterized them via the properties of the cosymplectic objects. The research narrowed down to the possible classes of manifolds in which the almost contact metric structure could lie and showed that the closed G_2-manifold admits almost contact metric 3-structure by constructing explicitly and characterizing when the almost contact metric 3-structure is cosymplectic. It was observed that 3-cosymplectic manifolds also known as hyper-cosymplectic manifolds were not considered by Todd's study. Indeed, we still do not know which odd dimensional manifolds support such structures. In a related work, Borman et'al [6] were concerned about the classification and existence of overtwisted contact structures in all dimensions. Their study was mooted from the background that there were no known general results concerning extensions of contact structures in dimension > 3. Since contact and almost contact structures are related, Borman et'al[6] were able to show that any closed, odd dimensional manifold with an almost contact structure admits a contact structure. In fact, the idea mostly revealed in their findings is that it is possible to study the geometric properties of a structure via the geometry of another structure epimorphic to it.

Galaev and Gokhman [14] studied the first integral of dynamical system with integrable linear connection and proved that an almost normal contact metric structures is a Sasakian structure. Accordingly, the Sasakian manifolds after their discovery by Sasaki[29] became very popular among the reseachers of almost contact metric spaces for the following reasons: there exists a big number of interesting and deep examples of Sasakian structures, also the Sasakian manifolds have very important and natural properties that overlap other almost contact structures, for example, normality of their metric connections and the satisfaction of the fundamental h-principle. Dually, the almost $\ddot{K}$hlerian spaces inherit many important structural properties of Sasakian spaces, this turns out to be very essential in cases when an almost contact metric space cannot in principle be a Sasakian space. 3-Sasakian structures have also appeared in supergravity and M-theory among other areas.

Puhle [28] studied almost contact metric 5 dimensional-manifold carrying one structure embedded with connection with torsion and proved that there exists a metric connection on the 5-dimensional almost contact metric manifolds compatible with almost contact structure. The findings were that the space of torsion tensor of a metric connection splits into ten $U(2)$-irreducible subspaces $W_1, W_2, ..., W_{10}$. Therefore, one can find 2^{10} classes of almost contact metric structure in 5-dimensional manifold according to the components of torsion tensors. This work also considered the normality property, that is $N_\phi + 2d\eta \otimes \xi = 0$ hence the integrability property of the manifold considered. Matzeu and Munteanu [22] studied vector cross products and almost contact structures and constructed almost contact metric structure induced by a 2-fold vector cross product on some classes of manifolds with G_2 structures. Chinea and Gonzales [11] in their work of classification of almost contact metric manifolds showed that their classification was achieved via the study of covariant derivative of the fundamental two form. Indeed, a space having the same symmetries as the covariant derivative of the fundamental two form was written and then this space was decomposed into twelve $U(n) \times 1$ irreducible components $C_1, C_2, ..., C_{12}$. Then there were 2^{12} invariant subspaces, each corresponding to a class of almost contact metric manifolds. They showed that there is a global 2-form and properties of the covariant derivative of this 2-form yielded 2^{12} classes of almost contact metric manifolds (see [3], [11]).

The existence of contact structures on closed odd dimensional manifolds is still questionable, however, Gromove [17] in the study of stable mapping of foliation into manifolds showed that contact structures on an open manifold obey an h-principle. Conventionally for (M^d, X) for d-odd a co-orientable contact manifold, then the tangent bundle of the manifold M can be expressed as: $TM = X \oplus \mathbb{R}$ and thus the structure group of the tangent bundle reduces to $U(n)$. Such a reduction of the structure group is called an almost contact structure on the manifold M. Thus a contact structure on M induces an almost contact structure. If M is an open manifold Gromove proved that the inclusion of the space of co-oriented contact structure on M into the space of almost contact structures on M is a weak homotopy property, Gromove [17]. The fundamental existence question in contact geometry concerns whether or not almost contact structures always come from contact structures. According to Gromove [17], for open manifold, it has been shown that all almost contact structures are homotopic to contact structures, but on closed manifolds much less is known. There had been a complete answer in dimensions 1 and 3; the 1-dimensional results are trivial and in 3-dimensional case, almost contact structures are equivalent to plane fields. All oriented and close 3-dimensional manifolds admit almost contact structures. Hence, every plane field is homotopic to a contact structure, but on closed manifolds much less is known, Gromov [17]. Later, Sasaki [29] studied differential manifolds with structures which are closely associated to almost contact structures and introduced a geometric structure related to almost contact structure. This geometry became known as Sasaki Geometry and has been studied extensively ever since, giving rise to fundamental geometric relationships among manifolds. Moreover, almost contact manifolds were earlier introduced by Gray [16] and described as an odd-dimensional manifold whose structure group of tangent bundle can be reducible to $U(n) \times 1$.

In 1970, Kuo [20] studied the almost contact 3-structure and found out that a product of a manifold with almost contact 3-structure and a straight line results to quarternionic structure. The work gave better findings as compared to the classical perspectives concerning contact structures. Kuo demonstrated the existence of a metric compatible with almost contact manifolds. The results were extended to more than 1 structure by redefining the notion of Sasaki[29] and introduced manifolds with Sasakian 3-structures which were also studied independently by other scholars(see [32], [33], [38]). Almost contact manifolds with 3-structures were introduced in order to give a structure of contact type that is similar to an almost quarternionic structure in the same way an almost contact structure is similar to an almost complex structure. Kuo [20] further showed that the 3-Sasakian geometry has some interesting topological implications. Using earlier results of Tachibana about the harmonic forms on compact Sasakian space(see Tachibana [34]), Kuo showed that odd Betti numbers up to the middle dimensions must be divisible by 4. In 1969, Takahashi [35] studied Sasakian manifold with pseudo-Riemannian metric and introduced an almost contact manifold equipped with associated metric. He studied Sasakian manifolds equipped with associated indefinite metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as ε-Sasakian almost contact metric manifolds and ε-Sasakian manifolds respectively.

From the aforementioned literature, it is worth noting that the study of almost contact 1,2,3-manifolds has been explored before by a number of Geometers to an extent (see for example [5],[20],[27],[28]). However, little known is the presence of an almost contact 4-structure on any odd dimensional manifold M^{5n+4}. In fact given 2 almost contact structures, Kuo [20] proposed a third almost contact structure on M^{4n+3} but did not ascertain the validity of the structure by proving the structure tensor properties. Moreover Tachibana and Yu [33] conjectured the non-existence of a fourth almost contact structure on any odd dimensional manifold and satisfying the anti-commutativity condition. This research provides a proof of the existence of the third almost contact structure and further constructs a fourth almost contact structure (ϕ_4, ξ_4, η_4) on the manifold $M^{5n+4} \cong (N^{4n+3} \otimes \mathbb{R}^d)$; $d|(2n+1)$ and $gcd(2,n) = 1$. Finally, the study explores the geometry of the submersion between the manifold carrying 4 structures and the one carrying 3 structures giving rise to new forms of Gauss, Weingarten, Codazzi and Ricci equations.

CHAPTER THREE

GEOMETRY OF A FOURTH ALMOST CONTACT STRUCTURE DEVELOPED FROM THE THREE ALMOST CONTACT STRUCTURES

3.1 Fundamental Principles

In this section we provide a survey of some preliminary results useful in the body of the work: Suppose M is a $(2n+1)$-dimensional differentiable manifold, (ϕ, ξ, η) is a field of endomorphisms of the tangent spaces TM as a (1,1)-tensor field, a vector field and a 1-form on M respectively. If the triple (ϕ, ξ, η) satisfies the three conditions:

$$\begin{aligned} \phi^2(Y_i) &= -(Y_i) + \eta(Y_i)\xi && (3.1) \\ \eta(\xi) &= 1 && (3.2) \\ \eta \circ \phi &= 0 && (3.3) \end{aligned}$$

for any $Y_i \in \Gamma(TM), i \in \mathbb{N}$ and non-singular vector ξ then the triple above is called an almost contact structure and M is called an almost contact manifold (c.f [36]).

An almost contact structure has many similarities to an almost complex one. Thus every almost contact structure has got an associated almost complex structure and their construction can be done through J_i's, the almost complex structures. Suppose a differentiable manifold admits almost contact 3-structure (ϕ_i, ξ_i, η_i), for all $i = 1, 2, 3$ satisfying:

$$\begin{aligned} \eta_i(\xi_j) = \eta_j(\xi_i) = 0 \\ \phi_i \xi_j = -\phi_j \xi_i = \xi_k \\ \eta_i \circ \phi_j = -\eta_j \circ \phi_i = \eta_k \\ \phi_i \phi_j - \xi_i \otimes \eta_j = -\phi_j \phi_i + \xi_j \otimes \eta_i = \phi_k \end{aligned}$$

for all exhaustive permutations (i, j, k) of $(1, 2, 3)$, then there exists three almost complex structures J_1, J_2, J_3 associated with each almost contact structures (c.f [20]). It is upon this foundation that our fourth almost contact structure will be constructed.

The next Theorem is a classification result:

Theorem 3.1.1. *An almost contact metric manifold with* (M, ϕ, ξ, η, g) *is:*

(i) Contact if $\Phi = d\eta$ *such that the volume form does not vanish.*

(ii) K-contact if $\Phi = d\eta$ *and* ξ *is killing.*

(iii) Quasi-Sasakian if $d\Phi = 0$ *and* M *is normal.*

(iv) Sasakian if $\Phi = d\eta$ *and* M *is normal.*

(V) Kenmotsu if $(\nabla_{Y_1}\phi Y_2) = -\eta(Y_2)\eta(Y_1) - g(Y_1, \phi Y_2)\xi$.

(Vi) Cosymplectic if $d\phi = 0$, $d\eta = 0$ *and* $\Phi(Y_1, Y_2) = g(\phi Y_1, Y_2)$.

where, $Y_1, Y_2 \in \Gamma(TM)$, *d the exterior differential operator and* Φ *a 2-form.[3]*

The parameters of the theorem 3.1.1 are the considerations upon which recent classification of manifolds have been based.

Now, given M, the $(2n + 1)$-dimensional manifold described above, carrying almost contact 1-structure. Then, the following results which hold on M will be usefull in the sequel:

Proposition 3.1.1. *Given a 1-structure* (ϕ, ξ, η) *on M, the almost contact manifold, then the following conditions hold:*

$$\phi(\xi) = 0 \tag{3.4}$$

$$\eta \circ \phi = 0 \tag{3.5}$$

$$rank(\phi) = 2n \tag{3.6}$$

Proof. Let $\xi \in TM$ be a non-singular vector field, then, given the prescribed ϕ, η, then

$$\begin{aligned} \phi^2(\xi) &= -\xi + \eta(\xi)\xi \\ &= -\xi + 1.\xi \\ \phi^2(\xi) &= 0 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} 0 &= \phi^2(\phi(\xi)) \\ &= -\phi(\xi) + \eta(\phi(\xi))\xi \end{aligned} \tag{3.8}$$

so we have

$$\phi(\xi) = \eta(\phi(\xi))\xi \tag{3.9}$$

From equation 3.7, it is clear that $\phi(\xi) = 0$ or $\phi(\xi)$ is a non-zero vector field whose image is zero. To the contrary, let $\phi(\xi)$ be a vector field: $0 \neq \phi(\xi) : \phi(\xi) \longrightarrow 0$. In this case $\eta(\phi(\xi))$ is not zero. If $\eta(\phi(\xi)) = 0$, then $\phi(\xi) = 0$ in equation 3.9 which is a contradiction to the assumption. By equation 3.9

$$\phi^2(\xi) = \phi(\phi(\xi)) = \phi(\eta(\phi(\xi))\xi) = \eta(\phi(\xi)).\phi(\xi) = \eta(\phi(\xi)).\eta(\phi(\xi)).\xi = \eta(\phi(\xi))^2.\xi$$

and we have a nontrivial $\phi^2(\xi)$ because $\eta(\phi(\xi))$ and ξ are non-zero. But this contradicts to the fact that $\phi^2(\xi) = 0$. Therefore we conclude that $\phi(\xi) = 0$ and equation 3.4 is proved.

Next, we have $\phi^2(Y_1) = -Y_1 + \eta(Y_1)\xi$, we get:
$\phi^3(Y_1) = \phi(\phi^2(Y_1)) = \phi(-Y_1 + \eta(Y_1)\xi) = \phi(-Y_1) + \phi(\eta(Y_1)\xi) = -\phi Y_1 + \phi(\eta(Y_1)\xi)$
for any vector Y_1. On the other hand, we write $\phi^3(Y_1)$ as:

$$\phi^3(Y_1) = \phi^2(\phi(Y_1)) = -\phi(Y_1) + \eta(\phi(Y_1))\xi$$

then,
$\eta(\phi(Y_1))\xi = \phi^3(Y_1) + \phi(Y_1) = -\phi(Y_1) + \eta(\phi(Y_1))\xi + \phi(Y_1) = \eta(\phi(Y_1))\xi = 0$
From previous result 3.7, we have: $\phi(\xi) = 0$. Therefore, $\eta \circ \phi = 0$: $Y_1 \in M$.

Finally, let $rank(\phi) = 2n$. Since $\phi(\xi) = 0$, it is clear that ϕ has dimension less than or equal to $2n$. Suppose there is another vector Y_1 of M such that $\phi(Y_1) = 0$. Then $\phi^2(Y_1) = \phi(\phi(Y_1)) = -Y_1 + \eta(Y_1)\xi$ implies that $Y_1 = \eta(Y_1)\xi$. □

Assuming that the manifold M is paracompact so that it allows a Riemannian metric tensor f' and a Riemannian metric f, then denoting and defining the convoluted tensor f as:

$f : f(Y_1, Y_2) = f'(\phi^2(Y_1), \phi^2(Y_2)) + \eta(Y_1)\eta(Y_2) = f[-Y_1 + \eta(Y_1)\xi, -Y_2 + \eta(Y_2)\xi] + \eta(Y_1)\eta(Y_2)$ implies that the relationship between the two tensors can be concretized. This implies that the primitive metric g can be expressed in terms of the said tensors. It is then clear that the next results holds:

Lemma 3.1.1. *Let $Y_1, Y_2 \in TM$ and*

$$f : f(Y_1, Y_2) = f'(\phi^2(Y_1), \phi^2(Y_2)) + \eta(Y_1)\eta(Y_2) = f[-Y_1 + \eta(Y_1)\xi, -Y_2 + \eta(Y_2)\xi] + \eta(Y_1)\eta(Y_2)$$

then,

$$f = \eta(Y_1) \tag{3.10}$$

for every vector field $Y_1 \in M$.

Proof. Let $Y_2 = \xi$. Then, by definition of f,

$$f : f(Y_1, \xi) = f'(\phi^2(Y_1), \phi^2(\xi)) + \eta(Y_1)\eta(\xi) = \eta(Y_1).$$

and the result follows. □

Proposition 3.1.2. *The almost contact manifold M above admits a primitive Riemannian metric tensor field g expressible in terms of f and with the property:*

$$g : g(\phi(Y_1), \phi(Y_2)) = g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2) \tag{3.11}$$

Proof. Let f be given and g be expressed as
$g(Y_1, Y_2) = \frac{1}{2}(f(Y_1, Y_2) + f(\phi Y_1, \phi Y_2) + \eta(Y_1)\eta(Y_2))$ with the same Riemannian metric f as $f : f(Y_1, \xi) = \eta(Y_1)$. We rewrite $g(\phi(Y_1), \phi(Y_2))$ as

$$f : f(\phi Y_1, \phi Y_2) = \frac{1}{2}(f(\phi Y_1, \phi Y_2) + f(\phi^2 Y_1, \phi^2 Y_2) + \eta(\phi Y_1)\eta(\phi Y_2)).$$

Since $\eta \circ \phi = 0$,

$$g : g(\phi Y_1, \phi Y_2) = \frac{1}{2}(f(\phi Y_1, \phi Y_2) + f(-Y_1 + \eta(Y_1)\xi, -Y_2 + \eta(Y_2)\xi))$$

$$= \frac{1}{2}(f(\phi Y_1, \phi Y_2) + f(Y_1, Y_2) - \eta(Y_2)f(Y_1, \xi) - \eta(Y_1)(f(\xi, Y_2)) + \eta(Y_1)\eta(Y_2)f(\xi, \xi))$$

$$= \frac{1}{2}(f(\phi Y_1, \phi Y_2) + f(Y_1, Y_2) - \eta(Y_2)\eta(Y_1) - \eta(Y_1)\eta(Y_2) + \eta(Y_1)\eta(Y_2))$$

$$= \frac{1}{2}(f(\phi Y_1, \phi Y_2) + f(Y_1, Y_2) - \eta(Y_2)\eta(Y_1))$$

$$= g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2)$$

□

Remark 1. *Since $\eta \circ \phi = 0$,*

$$g : g(\phi Y_1, Y_2) = g(\phi^2 Y_1, \phi Y_2) + \eta(\phi(Y_1))\eta(Y_2)$$

$$= g(\phi^2 Y_1, \phi Y_2)$$

$$= g(-Y_1 + \eta(Y_1)\xi, \phi Y_2)$$

$$= g(-Y_1, \phi Y_2) + \eta(Y_1) g(\xi, \phi Y_2)$$

$$= -g(Y_1, \phi Y_2)$$

because $g(\xi, \phi Y_2) = g(\phi\xi, \phi^2 Y_2) + \eta(\xi)\eta(\phi Y_2) = 0$. Hence, ϕ is a skew-symmetric tensor field with respect to the metric g. That is,

$$g(\phi Y_1, Y_2) + g(Y_1, \phi Y_2) = 0.$$

because $g(\xi, \phi Y_2) = g(\phi\xi, \phi^2 Y_2) + \eta(\xi)\eta(\phi Y_2) = 0$.

Hence, ϕ is a symmetric tensor field. In fact it is a skew-symmetric. That is

$$[g(\phi Y_1, Y_2) + g(Y_1, \phi Y_2)] = 0. \tag{3.12}$$

If M admits the aggregate (ϕ, ξ, η, g) and the conditions 3.1, 3.2, 3.3, then we say that M has an almost contact metric structure (ϕ, ξ, η, g) and (M, ϕ, ξ, η, g) is called an almost contact metric manifold.

3.2 The Construction of the fourth structure (ϕ_4, ξ_4, η_4) on $M^{5n+4} \cong N^{4n+3} \otimes \mathbb{R}^d$

The following result is necessary for the feasibility of a third structure.

Theorem 3.2.1. *Let* $\phi_1, \phi_2 \in T_{(1,1)}$*,* $\xi_1, \xi_2 \in TM$ *and* $\eta_1, \eta_2 \in TM^*$*. Suppose* (ϕ_1, ξ_1, η_1) *and* (ϕ_2, ξ_2, η_2) *are both almost contact structures obeying:*

$$\begin{aligned}
\phi_1\phi_2 + \phi_2\phi_1 &= \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1 \\
\phi_1\xi_2 + \phi_2\xi_1 &= 0 \\
\eta_1 \circ \phi_2 + \eta_2 \circ \phi_1 &= 0 \\
\eta_2(\xi_1) = 0& \\
\eta_1(\xi_2) = 0&
\end{aligned}$$

then the sets (ϕ_1, ξ_1, η_1) *and* (ϕ_2, ξ_2, η_2) *are said to admit an additional structure.*

We provide the proof to establish sufficiency:

Proof. Considering the possible combinations of the aggregates: set $\phi_3 = \phi_1\phi_2 - \eta_2 \otimes \xi_1 = -\phi_2\phi_1 + \eta_1 \otimes \xi_2$, $\eta_3 = \eta_1 \circ \phi_2 = -\eta_2 \circ \phi_1$ and $\xi_3 = \phi_1\xi_2 = -\phi_2\xi_1$. We can verify that the structure (ϕ_3, ξ_3, η_3) defines an almost contact 3-structure.

Assume that $X \in TM$, then $X \in M^{4n+3}$ which is a smooth manifold and so if ϕ_3 is properly choosen then $\phi_3^2 = -X + \eta_3 \otimes \xi_3$ holds. Now we need to show that $\eta_3(\xi_3) = 1$ and $\eta_3 \circ \phi_3 = 0$ in order for (ϕ_3, ξ_3, η_3) to qualify to be an almost contact structure on M^{4n+3}.

First, we show that $\eta_3(\xi_3) = 1$.

Let, $\eta_3 = -\eta_2 \circ \phi_1$ and $\xi_3 = \phi_1\xi_2$, $\eta_3(\xi_3) = -\eta_2 \circ \phi_1(\xi_3) = -\eta_2(\phi_1(\xi_3))$

But,

$$\phi_1(\xi_3) = \phi_1(\phi_1\xi_2) = \phi_1^2(\xi_2). \tag{3.13}$$

We have that $\phi_1^2 = -I + \eta_1 \otimes \xi_1$, substituting this in the equation 3.13 we have:

$$\begin{aligned}
\phi_1^2(\xi_2) &= -\xi_2 + \eta_1(\xi_2)\xi_1 \\
&= -\xi_2 + 0 \\
&= -\xi_2
\end{aligned}$$

So we have:

$-\eta_2(\phi_1\xi_3) = -\eta_2(-\xi_2) = \eta_2\xi_2 = 1.$

Thus,

$\eta_3(\xi_3) = 1$.

Next, we show that: $\phi_3\xi_3 = 0$

Let, $\xi_3 = -\phi_2\xi_1$,

then

$$\phi_3\xi_3 = \phi_3(-\phi_2\xi_1) = \phi_3(-\phi_2(X)\xi_1) \tag{3.14}$$

Also, $\phi_3 = \phi_1\phi_2 - \eta_2 \otimes \xi_1$, substituting this in the equation 3.14 we get:

$$\phi_3\xi_3 = \phi_1\phi_2(X) - \eta_2(X)\xi_1(-\phi_2(X)\xi_1) \tag{3.15}$$

Simplifying the equation 3.15 we have:

$$\begin{aligned} &= -\phi_1(\phi_2^2(X)\xi_1) - 0 \\ &= -\phi_1(\phi_2^2(\xi_1)) \end{aligned}$$

But,

$$\begin{aligned} \phi_2^2(\xi_1) &= -\xi_1 + \eta_2 \otimes \xi_1 \\ &= -\phi_1[-\xi_1 + \eta_2 \otimes \xi_1] \\ &= -\phi_1[-\xi_1 + \eta_2(X)\xi_1] \\ &= -\phi_1[-\xi_1 + \eta_2(\xi_2)\xi_1] : X = \xi_2 \\ &= -\phi_1[-\xi_1 + \xi_1] \\ &= 0. \end{aligned}$$

Also, assuming that $X = \xi_1$, then we have,

$$\begin{aligned} &= -\phi_1[-\xi_1 + \eta_2(\xi_1)\xi_1] \\ &= \phi_1\xi_1 \\ &= 0. \end{aligned}$$

Hence, $\phi_3\xi_3 = 0$, as required.

Finally, we show that, $\eta_3 \circ \phi_3 = 0$

Now, $\eta_3 \circ \phi_3(X) = \eta_3(\phi_3(X))$

Let, $\phi_3 = \phi_1\phi_2 - \eta_2\xi_1$. Then, for $\eta_3 = \eta_1 \circ \phi_2 = -\eta_2 \circ \phi_1$ and $X \in TM$.

$$\begin{aligned}
\eta_3(\phi_3(X)) &= \eta_3(\phi_1\phi_2(X) - \eta_2(X)\xi_1) \\
&= -\eta_2 \circ \phi_1(\phi_1\phi_2(X) - \eta_2(X)\xi_1) \\
&= -\eta_2 \circ \phi_1(\phi_1(\phi_2(X)) + \eta_2(X)\xi_1(\eta_2 \circ \phi_1) \\
&= -\eta_2(\phi_1^2(\phi_2(X)) + \eta_2(X)(\eta_2(\phi_1\xi_1)) \\
&= -\eta_2(\phi_1^2(\phi_2(X)) + 0 \\
&= -\eta_2[-\phi_2(X) - \eta_1(\phi_2(X))\xi_1] \\
&= \eta_2\phi_2(X) - \eta_1(\phi_2(X))\eta_2(\xi_1) \\
&= \eta_2 \circ \phi_2(X) - \eta_1(\phi_2(X)).0 \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

Therefore, $\eta_3 \circ \phi_3 = 0$. Hence, any two manifold-structures (ϕ_1, ξ_1, η_1) and (ϕ_2, ξ_2, η_2) define essentially the same almost contact 3-structure. In this sense, we say that such almost contact structures (ϕ_i, ξ_i, η_i), for all $(i = 1, 2, 3)$ defined on M is an almost contact 3-structure. □

Following the results of Tachibana and Yu [33], in our construction, starting with almost contact 3-structures, we construct a new structure (ϕ_4, ξ_4, η_4) such that
$\eta_i(\xi_4) \neq \eta_4(\xi_i) \neq 0,\ i = 1, 2, 3$, necessarily. The dimension of the manifold carrying the almost contact 4-structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3), (ϕ_4, ξ_4, η_4) must be of the form $5n + 4; gcd(2, n) = 1$: The following results are useful in our construction:

Theorem 3.2.2. *Let M^d be a d-dimensional almost contact manifold such that d divides $2n + 1$. Then, there exist local and global coordinates $(x_1, ..., x_n, y_1, ..., y_n, f)$ with respect to which $\eta_4 = df_i - \sum_{i=1}^{4} y_i dx_i)$ in M^d.*

Proof. Let U be some coordinate neighborhood, subset of M^d. Choose an open-ball in U transverse to ξ_i such that $d\eta_i$ is even dimensional and hence symmetrically symplectic in

U. Then, there exist some global coordinates $(x_1, ..., x_n, y_1, ..., y_n, f)$ such that $d\eta_4 = \sum dx_i \wedge dy_i : i = 1, .., 4$. Now $d(\eta_4 + \sum_{i=1}^4 y_i dx_i) = 0$ so that $\eta_4 + \sum_{i=1}^4 y_i dx_i = df_4$ for some functions f_4. Clearly, $\eta \wedge (d\eta)^n = df_4 \wedge dx_1 \wedge ... \wedge dx_n \wedge dy_1 \wedge ... \wedge dy_n \neq 0$. Hence, the volume form does not vanish in U. Therefore df_4 is independent of (dx_i, dy_i) and thus we consider x_i, y_i and f_4 as a coordinate system. □

Remark 2. *The results of theorem 3.2.2 shows that the volume form of an almost contact manifold does not vanish.*

Proposition 3.2.1. *Let $(M^{5n+4}, (\phi_i, \xi_i, \eta_i))$; $i = 1, 2, 3$ be an almost contact 4-structure. Let $f : M^{5n+4} \longrightarrow N^{4n+3} \otimes \mathbb{R}^d : d|2n + 1$. Assume that there exists an aggregate $[J_i] : i = 1, 2, 3$ of almost complex structures given by:*

$$J_1\big(X, f\frac{d}{dt}\big) = \big(\phi_1 X - f\xi_1, \eta_1(X)\frac{d}{dt}\big),\ J_2\big(X, f\frac{d}{dt}\big) = \big(\phi_2 X - f\xi_2, \eta_2(X)\frac{d}{dt}\big) \tag{3.16}$$
$$J_3\big(X, f\frac{d}{dt}\big) = \big(\phi_3 X - f\xi_3, \eta_3(X)\frac{d}{dt}\big)$$

where $X \in \Gamma(TM)$ and $f \in C^\infty(N^{4n+3} \otimes \mathbb{R})$. Let J_i; $i = 1, .., , 3$ be integrable, that is $[J_i, J_i] = 0$ so that (ϕ_i, ξ_i, η_i) is hypernormal. Suppose there exist another almost complex structure J_4 such that $J_4\big(X, f\frac{d}{dt}\big) = \big(\phi_4 X - f\xi_4, \eta_4(X)\frac{d}{dt}\big)$ and $[J_i, J_i] = 0$, then (ϕ_4, ξ_4, η_4) is an almost contact structure. Moreover if $J_3J_4 = J_1J_2J_4 = -J_1J_4J_2 = J_4J_1J_2 = J_4J_3$, then (ϕ_4, ξ_4, η_4) defines an almost contact structure whose field of endomorphism satisfies the anticommutativity condition with the other three.

Proof. The proof follows from the proof of Theorem 3.2.1. □

We now qualify our construction as follows:

Let (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) be almost contact 3-structures on M^{5n+4}. From Theorem 3.2.1, we see that:

$$\phi_1\phi_2 + \phi_2\phi_1 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1 = 0\ \phi_1\xi_2 + \phi_2\xi_1 = 0 \tag{3.17}$$

so that $\phi_1 = \phi_2\phi_3 - \eta_3 \otimes \xi_2 = -\phi_3\phi_2 + \eta_2 \otimes \xi_3$, $\phi_2 = \phi_3\phi_1 - \eta_1 \otimes \xi_3 = -\phi_1\phi_3 + \eta_3 \otimes \xi_1$ and $\phi_3 = \phi_1\phi_2 - \eta_2 \otimes \xi_1 = -\phi_2\phi_1 + \eta_1 \otimes \xi_2$. Similar descriptions can be given for ξ_i and

η_i according to the same result. We need to construct (ϕ_4, ξ_4, η_4) such that each of the respective tensors is expressed in terms of the first three above.

With obvious identifications, we see that $\exists$ some endomorphism constructible from $\phi_1,\ \phi_2,\ \phi_3$; which are pairwise anticommutative and thus:

$$\begin{aligned}&\phi_1\phi_2 + \phi_2\phi_1 + \phi_1\phi_3 + \phi_3\phi_1 + \phi_2\phi_3 + \phi_3\phi_2 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1 + \\ &\eta_1 \otimes \xi_3 + \eta_3 \otimes \xi_1 + \eta_2 \otimes \xi_3 + \eta_3 \otimes \xi_2 = 0 \end{aligned} \tag{3.18}$$

From 3.18, by exhausting the permutations of all the possible combinations, we have possible constructions for ϕ_4, as:

$$\begin{aligned}\phi_4 &= \phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 - \big(\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3\big) \\ &= -\big(\phi_2\phi_1 + \phi_3\phi_2 + \phi_1\phi_3\big) + \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_3 + \eta_3 \otimes \xi_1 \end{aligned} \tag{3.19}$$

Similarly,

$$\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1 = -\big(\phi_2\xi_1 + \phi_3\xi_2 + \phi_1\xi_3\big) \tag{3.20}$$

But

$$\eta_1 \circ \phi_2 + \eta_2 \circ \phi_1 + \eta_1 \circ \phi_3 + \eta_3 \circ \phi_1 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_2 = 0,$$

and $\eta_i(\xi_j) = \eta_j(\xi_i) = 0;\ i \neq j,\ \eta_i(\xi_i) = 1,\ \eta_i(\phi_i) = 0\ \forall i = 1, 2, 3$. So we need an appropriate η_4 from the construction such that the aggregate (ϕ_4, ξ_4, η_4) is an almost contact structure. By inspection, we can immediately see that

$$\eta_4 = \frac{1}{3}\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big) = -\frac{1}{3}\big(\eta_2 \circ \phi_1 + \eta_3 \circ \phi_2 + \eta_1 \circ \phi_3\big) \tag{3.21}$$

Theorem 3.2.3. *Let $M^{5n+4} \cong N^{4n+3} \otimes \mathbb{R}^d$ be given. Assume that the hyper-Khalerian condition on almost cosymplectic structures J_i results to some J_4 given by the proposition 3.2.1, then there exists the aggregate (ϕ_4, ξ_4, η_4) on $M^{5n+4} \cong N^{4n+3} \otimes \mathbb{R}^d$ and embedded on the hidden compartment.*

Proof. Recall that $\xi_1 = \phi_2\xi_3 - \phi_3\xi_2,\ \xi_2 = \phi_3\xi_1 - \phi_1\xi_3,\ \xi_3 = \phi_1\xi_2 - \phi_2\xi_1$. Let $\phi_4^2 = -I + \eta_4 \otimes \xi_4$, we show that $\eta_4(\xi_4) = 1,\ \phi_4\xi_4 = 0$ and $\eta_4 \circ \phi_4 = 0$ as follows:

$$\eta_4(\xi_4) = \frac{1}{3}\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1\big)$$
$$= \frac{1}{3}\big(\{\eta_1(\phi_2\xi_1) + \eta_1(\phi_2\xi_2) + \eta_1(\phi_2\xi_3)\} + \{\eta_2(\phi_3\xi_1) + \eta_2(\phi_3\xi_2) + \eta_2(\phi_3\xi_3)\} +$$
$$\{\eta_3(\phi_1\xi_1) + \eta_3(\phi_1\xi_2) + \eta_3(\phi_1\xi_3)\}\big)$$
$$= \frac{1}{3}\big(- \eta_1\xi_3 + \eta_1\xi_1 + \eta_2\xi_2 - \eta_2\xi_1 + \eta_3\xi_3 - \eta_3\xi_2\big) = \frac{1}{3}(3) = 1 \tag{3.22}$$

Next,

$$\phi_4\xi_4 = \big(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 - \big(\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3\big)\big)\big(\xi_1 + \xi_2 + \xi_3\big)$$
$$= \big(\phi_1\phi_2\xi_1 + \phi_1\phi_2\xi_2 + \phi_1\phi_2\xi_3 + \phi_2\phi_3\xi_1 + \phi_2\phi_3\xi_2 + \phi_2\phi_3\xi_3 + \phi_3\phi_1\xi_1 + \phi_3\phi_1\xi_2$$
$$+ \phi_3\phi_1\xi_3\big) - \big(\eta_2 \sum_{i=1}^{3}(\xi_i) \otimes \xi_1 + \eta_3 \sum_{i=1}^{3}(\xi_i) \otimes \xi_2 + \eta_1 \sum_{i=1}^{3}(\xi_i) \otimes \xi_3\big)$$
$$= \big(- \phi_1\xi_3 - \phi_2\xi_1 - \phi_3\xi_2\big) - \big(\sum_{i=1}^{3}(\xi_i)\big) = \big(\sum_{i=1}^{3}(\xi_i)\big) - \big(\sum_{i=1}^{3}(\xi_i)\big) = 0 \tag{3.23}$$

Finally,

$$\eta_4 \circ \phi_4 = \frac{1}{3}\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_4\big) = \frac{1}{3}\big(\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_4\big)\big)$$
$$= \frac{1}{3}\big\{\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1\big) -$$
$$\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3\big)\big\}$$
$$= \frac{1}{3}\big\{\eta_1(\phi_2\phi_2\phi_3) + \eta_2(\phi_3\phi_3\phi_1) + \eta_1(\phi_1\phi_1\phi_2)\big\} - \frac{1}{3}\big\{\eta_1\phi_2(\eta_2 \otimes \xi_1) + \eta_1\phi_2(\eta_3 \otimes \xi_2) +$$
$$\eta_1\phi_2(\eta_1 \otimes \xi_3) + \eta_2\phi_3(\eta_2 \otimes \xi_1) + \eta_2\phi_3(\eta_3 \otimes \xi_2) + \eta_2\phi_3(\eta_1 \otimes \xi_3) + \eta_3\phi_1(\eta_2 \otimes \xi_1) +$$
$$\eta_3\phi_1(\eta_3 \otimes \xi_2) + \eta_3\phi_1(\eta_1 \otimes \xi_3)\big\} \tag{3.24}$$

Applying a vector field $\xi_i \in \{\xi_1, \xi_2, \xi_3\}$ to equation 3.24, consider ξ_2 say, we have:

$$\frac{1}{3}\big\{\eta_1(\phi_2\phi_2\phi_3\xi_2) + \eta_2(\phi_3\phi_3\phi_1\xi_2) + \eta_1(\phi_1\phi_1\phi_2\xi_2)\big\} - \frac{1}{3}\big\{\eta_1\phi_2(\eta_2(\xi_2) \otimes \xi_1) + \eta_1\phi_2(\eta_3(\xi_2) \otimes \xi_2)$$
$$+ \eta_1\phi_2(\eta_1(\xi_2) \otimes \xi_3) + \eta_2\phi_3(\eta_2(\xi_2) \otimes \xi_1) + \eta_2\phi_3(\eta_3(\xi_2) \otimes \xi_2) + \eta_2\phi_3(\eta_1(\xi_2) \otimes \xi_3)$$
$$+ \eta_3\phi_1(\eta_2(\xi_2) \otimes \xi_1) + \eta_3\phi_1(\eta_3(\xi_2) \otimes \xi_2) + \eta_3\phi_1(\eta_1(\xi_2) \otimes \xi_3)\big\}$$
$$= \frac{1}{3}\big(\eta_1\phi_2(-\phi_2\xi_1)\big) - \frac{1}{3}\big(\eta_1(\phi_2\xi_1) + \eta_2(\phi_3\xi_1) + \eta_3(\phi_1\xi_1)\big)$$
$$= \frac{1}{3}\big(\eta_1(\phi_2\xi_3)\big) - \frac{1}{3}\big(- \eta_1\xi_3 + \eta_2\xi_2\big)\big) = \frac{1}{3}\big(\eta_1\xi_1 - \eta_2\xi_2\big) = 0 \tag{3.25}$$

Hence, the three structures, (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2) and (ϕ_3, ξ_3, η_3) defines essentially an almost contact 4-structure (ϕ_4, ξ_4, η_4). $\square$

Theorem 3.2.4. *Let $\mathbb{R}^{5n+4}$ for n is odd be the $(5n+4)$-dimensional Euclidean space. Consider $\Big\{x_i, y_i, z\Big\}$, $1 \leq i \leq 5n+4$ as coordinates of $\mathbb{R}^{5n+4}$ and define with respect to the natural field of frames a tensor field $\phi_4 \in TM \oplus TM^*$:*

$$\phi_4 = (\sum_{i=1}^{3}(X_i\partial/\partial x_i + Y_i\partial/\partial y_i) + Z\partial/\partial z)$$
$$= \sum_{i=1}^{3}(-1)^i(Y_i\partial/\partial x_i - X_i\partial/\partial y_i) + \sum_{i=1}^{3}(-1)^i Y_i y_i \partial/\partial z$$

Then the differential 1-form $\eta_4 = -\frac{1}{3}(\eta_2 \circ \phi_1 + \eta_3 \circ \phi_2 + \eta_1 \circ \phi_3)$ takes the form:

$$\eta_4 = \frac{1}{2}(\partial z - \sum_{i=1}^{n} y_i \partial x_i)$$

Additionally, the vector field $\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1$ becomes:

$$\xi_4 = 2(\partial/\partial z)$$

It is now easy to check that: $\phi_4^2 = -I + \eta_4 \otimes \xi_4$, $\eta_4(\xi_4) = 1$, $\eta_4 \circ \phi_4 = 0$ and $\phi_4\xi_4 = 0$ and rank of $\phi_4 = 5n+2$ for n is odd.

Finally, let $\sigma = 2\alpha$ where $\alpha = 5n+2$ for n is even, the metric g_M on $\mathbb{R}^{5n+4}$ will be given by:

$$g_M = \eta_4 \otimes \eta_4 + \frac{1}{4}\Big\{ - \sum_{i=1}^{\sigma/2}(\partial x_i \otimes \partial x_i + \partial y_i \otimes \partial y_i) + \sum_{i=\sigma/2+1}^{n}(\partial x_i \otimes \partial x_i + \partial y_i \otimes \partial y_i)\Big\}$$

with respect to the natural frame. Clearly, g_M is Riemannian metric with index σ and $(\phi_4, \eta_4, \xi_4, g_M)$ is normal in $\mathbb{R}^{5n+4}$. The connection ∇ on $\mathbb{R}^{5n+4}$ is involutive.

Proof. Follows from the proof given in Theorem 3.2.3 with some modification. $\square$

Corollary 1. *Let $(M^{5n+4}, g_M) \cong (N^{4n+3} \otimes \mathbb{R}^d, g_N)$ be the metric manifold, containing almost contact three structures (ϕ_i, ξ_i, η_i) for all $i = 1, 2, 3$ where ϕ_i are the three $(1,1)$ tensors, ξ_i the three vector fields and η_i the three 1-forms respectively whose constructions are given by Theorem 3.2.3. For an odd integer n, (M^{5n+4}, g_M) contains an almost contact* *structure (ϕ_4, ξ_4, η_4) constructible from (ϕ_i, ξ_i, η_i) for all $i = 1, 2, 3$ whose tensors are given below by:*

$$\phi_4 = \sum_{i=1,2,3,j=1,2,3} (\phi_i \phi_j) - \sum_{i=1,2,3,j=1,2,3} (\eta_j \otimes \xi_i)$$
$$= \sum_{i=1,2,3,j=1,2,3} -(\phi_j \phi_i) + \sum_{i=1,2,3,j=1,2,3} (\eta_i \otimes \xi_j)$$

$$\xi_4 = \sum_{i=1,2,3,j=1,2,3} (\phi_i \xi_j) = \sum_{i=1,2,3,j=1,2,3} -(\phi_j \xi_i)$$

$$\eta_4 = \frac{1}{3}\Big(\sum_{i=1,2,3,j=1,2,3} (\eta_i \circ \phi_j)\Big) = \frac{1}{3}\Big(\sum_{i=1,2,3,j=1,2,3} -(\eta_j \circ \phi_i)\Big) \tag{3.26}$$

Moreover, $\eta_i(\xi_4) = \eta_4(\xi_i) = 1$, for all $i = 1, 2, 3$.

Theorem 3.2.5. *From the construction of the almost contact metric structure $(\phi_4, \xi_4, \eta_4, g_M)$, given by the corollary 1 follows that, the computation of the Nijenhuis tensor associated with our tensors gives:*

$$N_{(1)}(Y_1, Y_2) = [\phi_4, \phi_4](Y_1, Y_2) + 2d\eta_4(Y_1, Y_2)\xi_4 \tag{3.27}$$
$$N_{(2)}(Y_1, Y_2) = (L_{\phi_4} Y_1 \eta_4)(Y_2) - (L_{\phi_4} Y_2 \eta_4)(Y_1) \tag{3.28}$$
$$N_{(3)}(Y_1, Y_2) = (L_{\xi_4} \phi_4)(Y_1) \tag{3.29}$$
$$N_{(4)}(Y_1, Y_2) = (L_{\xi_4} \eta_4)(Y_1) \tag{3.30}$$
$$\tag{3.31}$$

We will proof equation 3.27 and the proof of $3.28, 3.29, 3.30$ and 3.31 follow.

Proof. Let $Y_1, Y_2 \in T(M^{5n+4})$ and assume $Y_1 = X, Y_2 = Y$ then we have:

$$\begin{aligned}
2d\eta_4 &= X\eta_4(Y) - Y\eta_4(X) - \eta_4[X, Y] \\
&= Xg(\xi_4, Y) - Yg(\xi_4, X) - g(\xi_4[X, Y]) \\
&= g(\nabla_X \xi_4, Y) + g(\xi_4, \nabla_X Y) - g(\nabla_Y \xi_4, X) - g(\xi_4, \nabla_X Y - \nabla_Y X) \\
&= g(\xi_4, \nabla_X Y - \nabla_Y X - \nabla_X Y + \nabla_Y X) \\
&= g(\xi_4, 0) \\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
[\phi_4,\phi_4](X,Y) &= \phi_4^2[X,Y]-\phi_4[\phi_4X,Y]-\phi_4[X,\phi_4Y]+[\phi_4X,\phi_4Y]\\
&= -[X,Y]+\eta_4([X,Y])\xi_4-\phi_4[\phi_4X,Y]-\phi_4[X,\phi_4Y]+[\phi_4X,\phi_4Y]\\
&= -[X,Y]+g(\xi_4,[X,Y])\xi_4-\phi_4[\phi_4X,Y]-\phi_4[X,\phi_4Y]+[\phi_4X,\phi_4Y]\\
&= -[X,Y]+g(\xi_4[X,Y])\xi_4-\phi_4(\nabla_{\phi_4X}Y-\nabla_{Y\phi_4}X-\phi_4\nabla_YX)\\
&- \phi_4(\nabla_{X\phi_4}Y+\phi_4\nabla_XY-\nabla_{\phi_4}Y-\nabla_{\phi_4Y}X)+(\nabla_{\phi_4X}\phi_4)Y\\
&+ \phi_4\nabla_{\phi_4X}Y-(\nabla_{\phi_4Y}\phi_4)X-\phi_4\nabla_{\phi_4Y}X\\
&= -[X,Y]+g(\xi_4,[X,Y])\xi_4-\phi_4\nabla_{\phi_4X}Y+\phi_4^2\nabla_YX\\
&- \phi_4^2\nabla_XY+\phi_4\nabla_{\phi_4X}Y-\phi_4\nabla_{\phi_4Y}X\\
&= -[X,Y]-g(\xi_4,[X,Y])\xi_4-\nabla_YX+\eta_4(\nabla_YX)\xi_4\\
&+ \nabla_XY-\eta_4(\nabla_XY)\xi_4\\
&= -\nabla_XY+\nabla_YX+\nabla_XY-\nabla_YX+g(\xi_4,\nabla_XY-\nabla_YX-\nabla_XY)\xi_4\\
&= 0
\end{aligned}
$$

So

$$
N_{(1)}(Y_1,Y_2)=[\phi_4,\phi_4](Y_1,Y_2)+2d\eta_4(Y_1,Y_2)\xi_4=0. \tag{3.32}
$$

Appropriately, $N_{(2)}(Y_1,Y_2)=0$, $N_{(3)}(Y_1,Y_2)=0$ and $N_{(4)}(Y_1,Y_2)=0$. Hence the almost contact metric structure $(\phi_4,\xi_4,\eta_4,g_M)$ is normal. □

Proposition 3.2.2. *The almost contact metric structure (ϕ_4,ξ_4,η_4) satisfies the condition:*

$$
(\nabla_X\phi_4)Y=g_M(X,Y)\xi_4-\eta_4(Y)X\forall X,Y\in T(M^{5n+4}) \tag{3.33}
$$

Proof. Clearly, $\phi_4=\phi_1\phi_2+\phi_2\phi_3+\phi_3\phi_1-(\eta_2\otimes\xi_1+\eta_3\otimes\xi_2+\eta_1\otimes\xi_3)$,

$\xi_4=\phi_1\xi_2+\phi_2\xi_3+\phi_3\xi_1$,

$\eta_4=\frac{1}{3}(\eta_1\circ\phi_2+\eta_2\circ\phi_3+\eta_3\circ\phi_1)$

Substituting these in the equation 3.33 and considering that $\nabla_x\xi_4=-\phi_4X$, $Y=\xi_4$, we have

$$\begin{aligned}
(\nabla_X \phi_4)Y &= \nabla_X(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 - (\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3))Y \\
&= g_M(X,Y)(\phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1) \\
&- (\frac{1}{3}(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1))(X,Y)
\end{aligned}$$

where g_M is a metric on M^{5n+4}

$$\begin{aligned}
(\nabla_X \phi_4)Y &= \nabla_X \phi_1\phi_2 Y + \nabla_X \phi_2\phi_3 Y + \nabla_X \phi_3\phi_1 Y \\
&- \nabla_X Y \eta_2 \otimes \xi_1 - \nabla_X Y \eta_3 \otimes \xi_2 - \nabla_X Y \eta_1 \otimes \xi_3 \\
&= g_M(X,Y)\phi_1\xi_2 + g_M(X,Y)\phi_2\xi_3 + g_M(X,Y)\phi_3\xi_1 \\
&- \frac{1}{3}\eta_1 \circ \phi_2 XY - \frac{1}{3}\eta_2 \circ \phi_3 XY - \frac{1}{3}\eta_3 \circ \phi_3 XY \\
&= -\phi_4\phi_1\phi_2 - \phi_4\phi_2\phi_3 - \phi_4\phi_2\phi_1 \\
&+ \phi_4\eta_2 \otimes \xi_1 + \phi_4\eta_3 \otimes \xi_2 + \phi_4\eta_1 \otimes \xi_3 \\
&= g_M(X,\xi_4)\phi_1\xi_2 + g_M(X,\xi_4)\phi_2\xi_3 + g_M(X,\xi_4)\phi_3\xi_1 \\
&- \frac{1}{3}\eta_1\phi_2\xi_4(X) - \frac{1}{3}\eta_2\phi_3\xi_4(X) - \frac{1}{3}\eta_3\phi_1\xi_4(X) \\
&= 0.
\end{aligned}$$

Hence the required condition is satisfied. □

Remark 3. *From the results of proposition 3.2.2, we have observed that the aggregate*(ϕ_4, ξ_4, η_4) *is a fourth Sasakian structure.*

3.3 Geometry of Metric g_M of Tangent bundle $T(M^{5n+4})$

Proposition 3.3.1. *Let* g^I,g^{II},g^{III},g_N *be the positive definite metrics associated to the structures* (ϕ_1, ξ_1, η_1),....,(ϕ_4, ξ_4, η_4) *respectively in the differentiable manifold M of almost contact 4-structure. Then there exists another metric* g_M *of the structure such that if* $Y, Z \in T(M^{5n+4})$ *then for all* $i = 1, 2, 3, 4$.

$$g_M(Y,Z) = \frac{1}{5}\Big\{g_N(Y,Z) + \sum_{i=1}^{4} g_N(\phi_i(Y), \phi_i(Z)) + \eta_i(Y) + \eta_i(Z)\Big\} \tag{3.34}$$

Proof. Let g^I be the associated metric (ϕ_1, ξ_1, η_1) then is easy to see that g^{II},g^{III},g_N can be defined as

$$\begin{aligned} g^{II}(Y,Z) &= g^I(Y - \eta_2(Y)\xi_2, Z - \eta_2(Z)\xi_2) + \eta_2(Y)\eta_2(Z) \\ g^{III}(Y,Z) &= g^{II}(Y - \eta_3(Y)\xi_3, Z - \eta_3(Z)\xi_3) + \eta_3(Y)\eta_3(Z) \\ g_N(Y,Z) &= g^{III}(Y - \eta_4(Y)\xi_4, Z - \eta_4(Z)\xi_4) + \eta_4(Y)\eta_4(Z) \end{aligned} \tag{3.35}$$

it follows

$$5g_M(Y,Z) = g_N(Y,Z) + \sum_{i=1}^{4} g_N(\phi_i(Y), \phi_i(Z)) + \eta_i(Y) + \eta_i(Z) \tag{3.36}$$

From which

$$g_M(Y,Z) = \frac{1}{5}\Big\{g_N(Y,Z) + \sum_{i=1}^{4} g_N(\phi_i(Y), \phi_i(Z)) + \eta_i(Y) + \eta_i(Z)\Big\} \tag{3.37}$$

□

The equations 3.35, 3.36 and 3.37 hold for any vector in $T(M^{5n+4})$.

3.4 The Reeb Vector Field $\Big\{\xi_1, \xi_2, \xi_3, \xi_4\Big\}$

A nonparallel vector X on the manifold (M, g_M) is regarded as killing if $L_{Xg_M} = 0$. This condition can equivalently be given by: $\xi_{ij} + \xi_{ji} = 0$, the classical killing equation for covariant components $\xi_i = g_{ij}\xi^j$ of the vector field X. Thus generally, a field $X \in (M, g_M)$ is the killing vector field provided the angles between the field X and tangent vectors to every (oriented) geodesic in (M, g_M) are constant or vanishing along this geodesic. From the construction, we have: $\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1 = -\phi_2\xi_1 - \phi_3\xi_2 - \phi_1\xi_3$ and the next results hold;

Proposition 3.4.1. *Let D be the vertical distribution of $T(M^{5n+4})$ and $\xi_4 \in D$ given by $\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1$. Then ξ_4 is infinitesimal translation.*

Proof. Assume that every integral curve of any two $\xi_4's$ and $\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1$ is geodesic in (M^{5n+4}, g_M), we immediately have $L_{\xi_4 g_M}(\xi_4 X) = 0 : X \in M^{5n+4}$. This means that ξ_4 is killing and result follows. Dually let $X \in M^{5n+4}$ be arbitrary smooth vector field then the:

$$
\begin{aligned}
0 &= L_{\xi_4 g_M}(\xi_4 X) \\
&= \xi_4.g_M(\xi_4, X) - g_M([\xi_4, \xi_4], X) - g_M(\xi_4, [\xi_4, X]) \\
&= g_M(\nabla_{\xi_4}\xi_4, X) + g_M(\xi_4, \nabla_{\xi_4}, X) - g_M(\xi_4, [\xi_4, X]) \\
&= g_M(\nabla_{\xi_4}\xi_4, X) + g_M(\xi_4, \nabla_X \xi_4) \\
&= g_M(\nabla_{\xi_4}\xi_4, X) + \frac{1}{2} X.g_M(\xi_4, \xi_4) \\
&= 0.
\end{aligned}
$$

Hence the required result □

Proposition 3.4.2. *Let $\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1$ be Reeb vector field on (M, g_M). Since ξ_4 is an infinitesimal translation, it is killing and thus:*

(i) ξ_4 is a Jacobian vector field along the geodesic $f(t) : t \in \mathbb{R}$ in (M, g_M).

(ii) if $h(t) = \frac{1}{2} g_M(\xi_4(f(t)), \xi_4(f(t)))$ where $f(t)$ is geodesic on (M, g_M) then:

$$f''(t) = g_M(\nabla_{f'(t)}\xi_4, \nabla_{f'(t)}\xi_4) - g_M(R(\xi_4, f'(t))f'(t), \xi_4)$$

(iii) If x is the critical point of the length $g_M(\xi_4, \xi_4)^{\frac{1}{2}}$ of the field ξ_4 and $g_M(\xi_4, \xi_4) \neq 0$ then the integral trajection of ξ_4 passing through the point x is geodesic in (M^{5n+4}, g_M)

Proof. (i) Let R be the Ricci curvature on (M, g_M) and ∇ the usual Levi-civita connection. Suppose $f(t) : t \in \mathbb{R}$ is a geodesic curve of ξ_4 in (M, g_M) then $L_{\xi_4 g_M}(\xi_4, X) = 0$, for all $X \in \Gamma(TM)$. But in terms of R and ∇, we have that

$$\nabla^2_{f'(t)}\xi_4 + R(\xi_4, f'(t))f'(t) = 0 \tag{3.38}$$

where $f'(t)$ is the derivative of $f(t) : t \in \mathbb{R}$. By equation 3.38, we conclude that ξ_4 is Jacobi along $f(t)$

(ii) The proof of (ii) follows from (i) and the calculation:

$$\begin{aligned} h'(t) &= \frac{1}{2}\nabla_{f'(t)} g_M(\xi_4, \xi_4) \\ &= g_M(\nabla_{f'(t)}\xi_4, \xi_4) \\ &= g_M(\nabla^2_{f'(t)}\xi_4, \nabla_{f'(t)}\xi_4) - g_M(R(\xi_4, f'(t)f'(t), \xi_4)) \end{aligned}$$

(iii) The proof of (iii) follows from the proof provided in (ii) for $x \in \Gamma(TM)$.

$\square$

Proposition 3.4.3. *Let $\xi_4 \in \Gamma T(M^{5n+4})$, since $\xi_4 \in D$ and $TM = D \oplus H$, the vectors $\xi_1, \xi_2, \xi_3, \xi_4$ are killing vector fields and thus:*

$$(\nabla_{\xi_4}\eta_4)X + (\nabla_X \eta_4)\xi_4 = 0. \tag{3.39}$$

for $X \in TM^{5n+4}$.

Proof. Since we have, $\nabla_X \eta_4 = \phi_4 X$ and $\nabla_{\xi_4}\eta_4 = \phi_4 \xi_4$ then we can easily see that:

$\phi_4\xi_4 + \phi_4 X = 0$

implying that:

$\phi_4\xi_4 = -\phi_4 X.$

Substituting this to ϕ_4 we get:

$$\phi_1\phi_2\xi_4 + \phi_2\phi_3\xi_4 + \phi_3\phi_1\xi_4 - \eta_2 \otimes \xi_1\xi_4 - \eta_3 \otimes \xi_2\xi_4 - \eta_1 \otimes \xi_3\xi_4 \tag{3.40}$$

$$= -(-\phi_1\phi_2 X - \phi_2\phi_3 X - \phi_3\phi_1 X + \eta_2 \otimes \xi_1 X + \eta_3 \otimes \xi_2 X + \eta_1 \otimes \xi_3 X) \tag{3.41}$$

Take $\xi_4 = X = \xi_4$ and substitute in the equation 3.40 we get:

Equation: $3.40 = 3.41 = 0$ as required. Hence, ξ_4 is a killing vector field. $\square$

Proposition 3.4.4. *Let $(M^{5n+4}, \phi_4, \xi_4, \eta_4, g_M)$ be the d-dimensional Riemannian manifold. There is a field say V in M^{5n+4} commuting with ϕ_4 such that:*

$$(\nabla_X \phi_4)Y = \eta_4(Y)VX - g_M(VX, Y)\xi_4 \tag{3.42}$$

$\forall X, Y \in M^{5n+4}$ and ∇-Levi-civita connection. Moreover, the integral curves of ξ_4 are geodesics.

Proof. Let $X, Y \in T(M^{5n+4})$ be given and $\xi_4 \in D$. Then

$$(\nabla_X \phi_4)Y = \eta_4(Y)VX - g_M(VX, Y)\xi_4 \tag{3.43}$$

Replacing Y with ξ_4 in equation 3.43 and applying ϕ_4 we get:

$$\phi_4 V = \nabla \xi_4 \tag{3.44}$$

From equation 3.44 we get:

$$\nabla_{\xi_4}\xi_4 = \phi_4 V \xi_4 = V\phi_4\xi_4 = 0.$$

Moreover, $N^{(2)}(X, Y) = 0$ implies that:

$$d\eta_4(\phi_4 X, Y) = d\eta_4(\phi_4 Y, X)$$

Thus the linear transformation V on M is symmetric and since $\nabla_{\xi_4}\xi_4 = 0$, and thus the integral curves of ξ_4 are geodesics. $\square$

3.5 Geometric Relationship between (M^{5n+4}, g_M) and (N^{4n+3}, g_N)

From the previous section, we denote by g_M the metric compatible with
$M^{5n+4} \cong N^{4n+3} \otimes \mathbb{R}^d$ defined by:

$$g_M(X, Y) = \frac{1}{5}\Big\{g_N(X, Y) + \sum_{i=1}^{4} g_N(\phi_i(X), \phi_i(Y)) + \eta_i(X) + \eta_i(Y)\Big\} \tag{3.45}$$

where, M is a $5n + 4$ dimensional manifold and by g_N the metric compatible with N^{4n+3} defined by:

$$g_N(X, Y) = g^{III}(X - \eta_4(X)\xi_4, Y - \eta_4(Y)\xi_4) + \eta_4(X)\eta_4(Y)\forall X, Y \in M. \tag{3.46}$$

Submersions from a manifold to another are useful for comparing geometric structures between them. A differentiable map $F : (M, g_M) \to (N, g_N)$ between the Riemannian manifolds (M, g_M) and (N, g_N) is called isometric immersion (submanifold) if F_* is injective and

$$g_N(F_*X, F_*Y) = g_M(X, Y) \tag{3.47}$$

for $X, Y \in TM$ and F_* a derivative map.

A smooth map $F : (M, g_M) \to (N, g_N)$ is called a submersion if F_* is onto and satisfies equation 3.47, for vector fields X and Y tangent to the horizontal space $(kerF_*)^{\perp}$.
Let $F : (M^{5n+4}, g_M) \to (N^{4n+3}, g_N)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min(5n + 4, 4n + 3)$ for odd n, where the dimension of $M = 5n + 4$ and dimension of $N = 4n + 3$, then the kernel of the space F^* is denoted by $kerF_*$. Consider the orthogonal complementary space $\mathcal{H} = (kerF_*)^{\perp}$ to $kerF_*$. Then, the tangent bundle of M^{5n+4} has the following decomposition:

$$T(M^{5n+4}) = KerF_* \oplus \mathcal{H}$$

Similarly, we consider the complementary orthogonal space $(rangeF_*)^{\perp}$ to range F_* in the tangent bundle $T(N^{4n+3} \times \mathbb{R})$. Since, $rankF < min(5n + 4, 4n + 3)$, we always have that $(rangeF_*)^{\perp} \neq 0$. Thus $T(N^{4n+3} \times \mathbb{R})$ has the following decomposition:

$$T(N^{4n+3} \times \mathbb{R}) = (rangeF_*) \oplus (rangeF_*)^{\perp}.$$

The set of equations that describe the relationships between invariant quantities on the empirical submanifolds and the base manifold when the Riemannian connection is used, are expressed by the Gauss' formulae, Weingartens' formulae and the equations of Gauss, Codazzi and Ricci. In this section we extend these equations to Riemannian submersions between M^{5n+4} and N^{4n+3}.

To do this, we recall the pullback connection along a map and assume that
$\|F_*\|^2 = rankF$. Then we define Gauss formula for the map using the second fundamental form of Riemannian map. We also obtain Weingarten formula for the map using the linear connection $\nabla^{F\perp}$ in $(F_*(TM))^{\perp}$. From the formula of Gauss-Weingarten, we extend to Gauss, Ricci and Codazzi equations for submersions. This results may be useful in the sequel.

Proposition 3.5.1. *Let $F : (M^{5n+4}, g_M) \to (N^{4n+3}, g_N)$ be a submersion between M^{5n+4} and N^{4n+3} then the following will equivalently hold:*

(i) F is Riemannian at $p_1 \in TM$ and thus at every $p \in M$.

(ii) Π_{p_1} is a projection.

(iii) Π'_{p_1} is a projection.

Proof. Since (M^{5n+4}, g_M) and (N^{4n+3}, g_N) are compatible with g_M with respect to g_N, the map $F : M \to N$ is Riemannian if there exists the adjoint map $*F_*$ of F_* characterized by:

$$g_M(X, *F_{*p_1}Y) = g_N(F_{*p_1}, Y)$$

for some $X \in T_{p_1}M$ and $Y \in T_{F(p_1)}N$ and $p_1 \in M$. Additionally, F is smooth between the manifolds M and N, thus we can define linear transformation:

$$\begin{aligned} \Pi_{p_1} : T_{p_1}M \to T_{p_1}M; \Pi_{p_1} &= *F_{*p_1} \circ F_{*p_1} \\ \Pi'_{p_1} : T_{p_2}N \to T_{p_2}N; \Pi'_{p_1} &= F_{*p_1} \circ *F_{*p_1}. \end{aligned}$$

Hence, $\Pi_{p_1} \circ \Pi_{p_1} = \Pi_{p_1}$ and $\Pi'_{p_1} \circ \Pi'_{p_1} = \Pi'_{p_1}$. So both Π_{p_1} and Π'_{p_1} are projections and the results above is completely characterized. □

3.6 Gauss-Weingarten Formulae for α-Rotated Submersion between M^{5n+4} and N^{4n+3}

Definition 3.6.1. *For a smooth map $F : (M^{5n+4}, g_M) \longrightarrow (N^{4n+3}, g_N)$. Let $F \twoheadrightarrow \Pi$ where Π is any given projection on $\Gamma(M^{5n+4})$, then for any $\alpha \in M^{5n+4}$, we have $\Pi = \alpha F$ and call Π, the α-rotated submersion.*

Let $\Pi : M \to N$ be the α-rotated submersion between (M^{5n+4}, g_M) and (N^{4n+3}, g_N). Let $p_2 = \Pi(p_1)$ for $p_1 \in M$. Suppose $\overset{N}{\nabla}$ is a Levi-Civita connection on N^{4n+3}, for $X \in \Gamma(TM)$ and $V \in \Gamma(TN)$, we have:

$$\overset{N}{\nabla} X(V \circ \Pi) = \nabla^N_{\Pi_* X} V \tag{3.48}$$

where $\Pi^{-1}TN$ defines the pullback bundle having fibres $(\Pi^{-1}TN)_p = T_{\Pi(p)}N$ for $p \in M$. $Hom(TM, \Pi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. The second fundamental form of Π is given by:

$$(\nabla\Pi_*)(X,Y) = \overset{N}{\nabla^{\Pi}}{}_X\Pi_*(Y) - \Pi_*(\nabla^M_X Y) \tag{3.49}$$

$X, Y \in \Gamma T(M^{5n+4})$. This form is symmetric. In addition $(\nabla\Pi_*)(X,Y) \in \Gamma((ker\Pi_*)^{\perp})$, for $X, Y \in \Gamma T(M^{5n+4})$, hence it lacks components in range Π_*. The following results thus hold.

Proposition 3.6.1. *Let $\Pi : M \to N$ be the Riemannian α-rotated described above. Then,*

$$g_N((\nabla\Pi_*)(X,Y), (\Pi_*(Z))) = 0 \forall X, Y, Z \in \Gamma(ker\Pi_*^{\perp}) \tag{3.50}$$

Proof. Clearly, $(\nabla\Pi_*)(X,Y) \in \Gamma((range\Pi_*)^{\perp}) \in \Gamma((ker\Pi_*)^{\perp})$, for $X, Y, Z \in TM$. Hence, given any point $p \in M^{5n+4}$, we can write:

$$\overset{N}{\nabla^{\Pi}}{}_X\Pi_*(Y)(p) = \Pi_*(\nabla^M_X Y)(p) + (\nabla\Pi_*)(X,Y)(p) \forall X, Y \in \Gamma(ker\Pi_*)^{\perp} \tag{3.51}$$

where $\overset{N}{\nabla^{\Pi}}{}_X\Pi_*(Y) \in T_{\Pi}(p)N$, $\Pi_*(\nabla^M_X Y)(p) \in \Pi_{*p}(T_pM)$
and $(\nabla\Pi_*)(X,Y)(p) \in (\Pi_{*p}(T_pM))^{\perp}$. $\square$

Next, Let $\Pi : M^{5n+4} \to N^{4n+3}$ be a Riemannian submersion. We define $\mathcal{T}$ and $\mathcal{A}$ by:

$$\mathcal{A}_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F \tag{3.52}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F \tag{3.53}$$

where $E, F \in (M^{5n+4})$ and ∇^M is the levi-civita connection on g_M such that $F = \frac{1}{\alpha}\Pi$ and $E = \frac{1}{\alpha'}\Pi$ $\forall \alpha, \alpha' \in M^{5n+4}$.
From $T(M^{5n+4}) = ker\Pi_* \oplus \mathcal{H}$, we see that, $\Pi_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$, hence $\mathcal{T}$ and $\mathcal{A}$ are vertical and horizontal respectively. Now $\mathcal{T}$ satisfies,

$$T_U W = T_W U \forall U, W \in \Gamma(ker\Pi_*).$$

Again, from equation 3.52 and 3.53 we have:

$$\nabla^M_V W = \mathcal{T}_V W + \overset{\neg}{\nabla}_V W \tag{3.54}$$

$$\nabla^M_V X = \mathcal{H}\nabla^M_V X + \mathcal{T}_V X \tag{3.55}$$

$$\nabla^M_X V = \mathcal{A}_X V + \mathcal{V}\nabla^M_X V \tag{3.56}$$

$$\nabla^M_X Y = \mathcal{H}\nabla^M_X Y + \mathcal{A}_X Y \tag{3.57}$$

$$\tag{3.58}$$

for all $X, Y \in \Gamma((ker\Pi_*)^{\perp})$ and $V, W \in \Gamma(ker\Pi_*)$ where $\overset{\neg}{\nabla} = \mathcal{V}\nabla^M_V W$. Let ∇^N denote both the levi-civita connection of (g_N) and its pullback along Π. Then $\nabla^{\Pi\perp}$ linear on $(\Pi_*(TM))^{\perp}$ and $\nabla^{\Pi\perp} g_N = 0$.

Proposition 3.6.2. *Let $\Pi : M^{5n+4} \to N^{4n+3}$ be an α-rotated submersion. Then the map defined and denoted by S_V as:*

$$\nabla^N_{\Pi_* X} V = -S_V \Pi_* X + \nabla^{\Pi\perp}_X V \tag{3.59}$$

where $S_V \Pi_* X$ is the tangential component of $\nabla^N_{\Pi_* X} V$ is symmetric linear transformation.

Proof. It has been gotten from the pullback connection of ∇^N, thus at $p_1 \in M^{5n+4}$, we see that:

$\nabla^N_{\Pi_* X} V(p_1) \in T_{\Pi(p_1)} N$, $S_V \Pi_* X(p_1) \in \Pi_{*p_1}(T_{p_1} M)$ and $\nabla^{\Pi\perp}_X V(p_1) \in (\Pi_{*p_1}(T_{p_1} M))^{\perp}$. Clearly $S_V \Pi_* X$ is biliniear in V and $\Pi_* X$ and $S_V \Pi_* X$ at p_1 depend on V_{p_1} and $\Pi_{*p_1} X_{p_1}$. Computing directly, we get:

$$g_N(S_V \Pi_* X, \Pi_* Y) = g_N(V, (\nabla \Pi_*)(X, Y)) \tag{3.60}$$

for $X, Y \in \Gamma(ker\Pi_*)^{\perp}$ and $V \in \Gamma(range\Pi_*)^{\perp}$. Since $(\nabla \Pi_*)$ is symmetric, it follows that S_V is a symmetric transformation of range Π_*. □

Remark 4. *: The equations* 3.51 *is Gauss formula and equations* 3.55, 3.56, 3.57 *and* 3.58 *are weingarten equations for $\Pi : M^{5n+4} \to N^{4n+3}$*

Figure 3.1 shows that α-rotated submersion between M^{5n+4} and $N^{4n+3} \otimes \mathbb{R}^d$. The shaded regions are mapped isometrically to each other by Π and the unshaded regions are independent of each other.

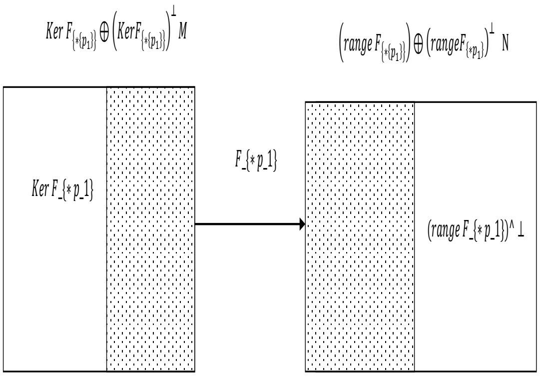

Figure 3.1: An α-rotated submersion between M and N

3.7 Gauss and Codazzi equations of submersion F between M^{5n+4} and N^{4n+3}

Let $F : M^{5n+4} \to N^{4n+3}$ the α- rotated submersion described in the previous section. Consider a linear transformation given and define by:

$\Pi^{\lambda}_{*p_1} : (ker\Pi_*)^{\perp}(p_1), g_{Mp_1}((ker\Pi_*)^{\perp}(p_1)) \to (range\Pi_*(p_2), g_{Np_2}(range\Pi_{*p_2})$. The adjoint of Π^{λ}_* is denoted by $*\Pi^{\lambda}_*$ and by $*\Pi_{*p_1}$ the adjoint of Π_{*p_1}. Now

$\Pi_{*p_1} : (T_{p_1}M, g_{Mp_1}) \to (T_{p_2}N, g_{Np_2})$. Then the linear transformation:

$(*\Pi_{*p_1})^{\lambda} : range\Pi_*(p_2) \to (ker\Pi_*)^{\perp}(p_1)$ defined by $(*\Pi_{*P_1})^{\lambda}Y = *\Pi_{*p_1}Y$ such that

$Y \in \Gamma(range\Pi_{*p_1})$, $p_2 = \Pi(p_1)$ is an epimorphism and $(\Pi^{\lambda}_{*p_1})^{-1} = (*\Pi_{*p_1})^{\lambda} = *(\Pi^{\lambda}_{*p_1})$.

Now suppose $\alpha \in M^{5n+4}$ is given and Π is the α-rotated submersion described, then there exists some meromorphism C which is isometrically mapped to Π that is $\mathsf{C} \twoheadrightarrow \Pi$, we use this C in the sequel:

Recall Gauss and Weingarten formulas 3.51 and 3.59 respectively. From them we have the

Ricci curvature in terms of $\mathtt{C}$ as:

$$\begin{aligned} R^N(\mathtt{C}_*\xi_i, \mathtt{C}_*\xi_j)\mathtt{C}_*Z &= -S_{(\nabla\mathtt{C}_*)(Y,Z)}\mathtt{C}_*X + S_{(\nabla\mathtt{C}_*)(X,Z)}\mathtt{C}_*Y \\ &+ \mathtt{C}_*(R^M(X,Y)Z) + (\nabla_X(\nabla\mathtt{C}_*))(Y,Z) \\ &- (\nabla_Y(\nabla\mathtt{C}_*))(X,Z)\forall i \neq j. \end{aligned} \tag{3.61}$$

$\forall X, Y, Z \in \Gamma(ker\mathtt{C}_*)^\perp$ where R^M, R^N represent the curvature tensors of ∇^M and ∇^N respectively the metric connection on M and N. Moreover, $(\nabla_X(\nabla\mathtt{C}_*))(Y,Z)$ is defined by:

$$\nabla_X(\nabla\mathtt{C}_*)(Y,Z) = \nabla_X^{\mathtt{C}\perp}(\nabla\mathtt{C}_*)(Y,Z) - (\nabla\mathtt{C}_*)(Y, \nabla_X^M Z) \tag{3.62}$$

From equation 3.61, for any vector $J \in \Gamma((ker\mathtt{C}_*)^\perp)$, we have:

$$\begin{aligned} g_N(R^N(\mathtt{C}_*X, \mathtt{C}_*Y)\mathtt{C}_*Z, \mathtt{C}_*J) &= g_M(R^M(X,Y)Z, J) \\ &+ g_N((\nabla\mathtt{C}_*)(X,Z), (\nabla\mathtt{C}_*)(Y,J)) \\ &- g_N((\nabla\mathtt{C}_*)(Y,Z), (\nabla\mathtt{C}_*)(X,J)) \end{aligned} \tag{3.63}$$

Taking the $\Gamma(range\mathtt{C}_*^\perp)$ in equation 3.61 we:

$$(R^N(\mathtt{C}_*X, \mathtt{C}_*Y)\mathtt{C}_*Z)^\perp = (\nabla_X(\nabla\mathtt{C}_*))(Y,Z) - (\nabla_Y(\nabla\mathtt{C}_*))(X,Z) \tag{3.64}$$

The equations 3.63 and 3.64 are the Gauss and Codazzi equations respectively for
$\mathtt{C} : M^{5n+4} \to N^{4n+3} \otimes \mathbb{R}^d$.

Next, Let $X, Y \in TM$ and $V^\perp \in \Gamma(range\mathtt{C}_*)$, the curvature tensor field $R^{\mathtt{C}\perp}$ of the sub-bundle $(range\mathtt{C}_*)^\perp$ is defined by:

$$R^{\mathtt{C}\perp}(\mathtt{C}_*(X), \mathtt{C}_*(Y))V = \nabla_X^{\mathtt{C}\perp}\nabla_Y^{\mathtt{C}\perp}V - \nabla_Y^{\mathtt{C}\perp}\nabla_X^{\mathtt{C}\perp}V - \nabla_{[X,Y]}{}^{\mathtt{C}\perp} \tag{3.65}$$

Then using Gauss-Weingarten equation 3.61, we obtain:

$$\begin{aligned} R^N(\mathtt{C}_*(X), \mathtt{C}_*(Y))V &= R^{\mathtt{C}\perp}(\mathtt{C}_*(X), \mathtt{C}_*(Y))V - \mathtt{C}_*(\nabla_X^M * \mathtt{C}_*(S_V\mathtt{C}_*(Y))) \\ &+ S_{\nabla_X^{\mathtt{C}\perp}V}\mathtt{C}_*(Y) + \mathtt{C}_*(\nabla_Y^M * \mathtt{C}_*(S_V\mathtt{C}_*(X))) \\ &- S_{\nabla_X^{\mathtt{C}\perp}V}\mathtt{C}_*(X) - (\nabla\mathtt{C}_*)(X, *\mathtt{C}_*(S_V\mathtt{C}_*(Y))) \\ &+ (\nabla\mathtt{C}_*)(Y, *\mathtt{C}_*(S_V\mathtt{C}_*(X))) - S_V\mathtt{C}_*([X,Y]) \end{aligned} \tag{3.66}$$

where,

$$\mathsf{C}_*([X,Y]) = \overset{N}{\nabla^{\mathsf{C}}}_X \mathsf{C}_*(Y) - \overset{N}{\nabla^{\mathsf{C}}}_Y \mathsf{C}_*(X).$$

Indeed for some $\mathsf{C}_*(Z) \in \Gamma(range\mathsf{C}_*)$, we have:

$$\begin{aligned} g_N(R^N(\mathsf{C}_*(X), \mathsf{C}_*(Y))V, \mathsf{C}_*(Z)) &= g_N((\tilde{\nabla}_Y S)_V \mathsf{C}_*(X), \mathsf{C}_*(Z)) \\ &- g_N((\tilde{\nabla}_X S)_V \mathsf{C}_*(Y), \mathsf{C}_*(Z)) \end{aligned} \tag{3.67}$$

where,

$$(\tilde{\nabla}_X S)_V \mathsf{C}_*(Y) = \mathsf{C}_*(\nabla^M_X * \mathsf{C}_*(S_V \mathsf{C}_*(Y))) - S_{\nabla^{\mathsf{C}\perp}_X V} \mathsf{C}_*(Y) - S_V \mathsf{C} \overset{N}{\nabla^{\mathsf{C}}}_X \mathsf{C}_*(Y)$$

where C denotes the projection meromorphism on the range C_*. Dually, for $W \in \Gamma(range\mathsf{C}_*^{\perp})$, we obtain,

$$\begin{aligned} g_N(R^N(\mathsf{C}_*(X), \mathsf{C}_*(Y))V, W) &= g_N(R^{\mathsf{C}\perp}(\mathsf{C}_*(X), \mathsf{C}_*(Y))V, W) - g_N((\nabla \mathsf{C}_*)(X, *\mathsf{C}_*(S_V \mathsf{C}_*(Y))), W) \\ &+ g_N((\nabla \mathsf{C}_*)(Y, *\mathsf{C}_*(S_V \mathsf{C}_*(X))), W) \end{aligned} \tag{3.68}$$

Using Gauss-Weingarten equation 3.61 , we obtain:

$$g_N((\nabla \mathsf{C}_*)(X, *\mathsf{C}_*(S_V \mathsf{C}_*(Y))), W) = g_N(S_W \mathsf{C}_*(X), S_V \mathsf{C}_*(Y)) \tag{3.69}$$

Since S_V is self adjoint, we get:

$$g_N((\nabla \mathsf{C}_*)(X, *\mathsf{C}_*(S_V \mathsf{C}_*(Y))), W) = g_N(S_V S_W \mathsf{C}_*(X), \mathsf{C}_*(Y)) \tag{3.70}$$

using equation 3.69 and 3.70 we arrive at:

$$\begin{aligned} g_N(R^N(\mathsf{C}_*(X), \mathsf{C}_*(Y))V, W) &= g_N(R^N(\mathsf{C}_*(X), \mathsf{C}_*(Y))^{\perp} V, W) \\ &+ g_N([S_W, S_V]\mathsf{C}_*(X), \mathsf{C}_*(Y)) \end{aligned} \tag{3.71}$$

where $[S_W, S_V] = S_W S_V - S_V S_W$. The last equation 3.71 is the Ricci equation for α-rotated submersion $\Pi : M^{5n+4} \to N^{4n+3} \otimes \mathbb{R}^d$

CHAPTER FOUR

SUMMARY OF FINDINGS, CONCLUSION AND RECOMMENDATIONS

4.1 Summary of Findings

The study was set up with an objective of characterizing certain geometric aspects of a class of almost contact structures on smooth metric manifold. This has been done in a number of steps by considering the specific geometric aspects. Starting from the background that Kuo [20] had proposed the existence of a third almost contact structure on a general smooth manifold N^{4n+3}, we provided a proof to the supposition in order to validate its existence by showing that: $\phi_3^2 = -I + \eta_3 \otimes \xi_3,\ \eta_3(\xi_3) = 1,\ \eta_3 \circ \phi_3 = 0$. The aggregate (ϕ_3, ξ_3, η_3) was expressed on a linear combination of the first two structures: (ϕ_1, ξ_1, η_1) and (ϕ_2, ξ_2, η_2). We then constructed the almost contact 4-structure (ϕ_4, ξ_4, η_4) expressed as permutation of linear combination of the first 3-structures on $(N^{4n+3} \otimes \mathbb{R}^d)$. These tensors were found to be of the form:

$$\begin{aligned}\phi_4 &= \sum_{i=1,2,3,j=1,2,3} (\phi_i \phi_j) - \sum_{i=1,2,3,j=1,2,3} (\eta_j \otimes \xi_i) \\ &= \sum_{i=1,2,3,j=1,2,3} -(\phi_j \phi_i) + \sum_{i=1,2,3,j=1,2,3} (\eta_i \otimes \xi_j)\end{aligned}$$

$$\xi_4 = \sum_{i=1,2,3,j=1,2,3} (\phi_i \xi_j) = \sum_{i=1,2,3,j=1,2,3} -(\phi_j \xi_i)$$

$$\eta_4 = \frac{1}{3}\Big(\sum_{i=1,2,3,j=1,2,3} (\eta_i \circ \phi_j)\Big) = \frac{1}{3}\Big(\sum_{i=1,2,3,j=1,2,3} -(\eta_j \circ \phi_i)\Big) \tag{4.1}$$

Then using an α-rotated submersion $\Pi : M^{5n+4} \longrightarrow N^{4n+3} \otimes \mathbb{R}^d : d|2n+1$ we were able to study the geometric relationships between the two manifolds via the submanifolds giving rise to a new form of Gauss, Weingarten, Gauss-Codazzi and Ricci equation incorporating the submersion.

4.2 Conclusion

In conclusion we mention that the objectives were all fully achieved. Indeed we established that there exists an intrinsic relationship between the manifolds M^{5n+4} and $N^{4n+3} \otimes \mathbb{R}^d$ since (ϕ_4, ξ_4, η_4) is embedded on $\mathbb{R}^d$. Furthermore the Reeb vectors obtained were killing giving rise to geodesic curves defined on M^{5n+4}.

4.3 Recommendation

The classification and study of geometry of high dimensional manifolds still remains open. We recommend an attempt of the following for further research.

1. The spacelike hypersurface on (M^{5n+4}, g_M).

2. The existence of more than 4 almost contact structures on M^d where $d > 5n + 4$ for an odd n.

3. The principles of open book decomposition on (M^{5n+4}, g_M).

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