DYNAMICS OF A FISHERY MODEL WITH ALLEE EFFECT IN POPULATION GROWTH

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DECLARATION

This thesis is my original work prepared with no other than the indicated sources and support and has not been presented elsewhere for a degree or any other award.

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CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance of Masinde Muliro University of Science and Technology a thesis entitled "**Dynamics of A fishery Model With Allee Effect in Population Growth**".

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DEDICATION

This work is dedicated to my family; my daughter Rebecca, sons Walter and Riemann, wife Everline, my parents and late grandmother. You have been a great inspiration in my life.

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ABSTRACT

Sustainable exploitation of renewable resources such as marine fish has been a concern to both ecologists and economists with threats of species extinction due to over - exploitation a stark reality. Fish exhibits co-operative and social characteristics among individuals which increases its survival through predator surveillance and enhanced spawning. Overfishing compromises this con - specific interactions and socialism. In this study, a mathematical model of a fishery with an Allee effect in the population growth equation is developed. This is achieved by formulation of a set of ordinary differential equations governing the evolution in the population growth, harvesting effort and the stock market price. The model is aggregated to reduce its dimensions from a system of three equations to a system of two equations since market price is taken to evolve faster. Four equilibrium points are obtained, and analysed using local stability and local bifurcation. Simulation of the model for equilibrium points with zero harvesting is performed using MATLAB while further stability analysis, at three different values of the threshold population is carried out yielding three different cases of equilibrium solutions; Stable equilibrium, co-existence of three positive equilibria two being saddles and the other being stable and the co-existence of three positive equilibria with two being stable separated by a saddle. The stable equilibrium corresponds to the case of a fishery with the fish stock at high levels with a small economic activity, the harvesting is artisanal with low economic returns. The co-existence of three positive equilibria, two being saddles and the other being stable corresponds to a fishery where, fishery management practices of monitoring, control and surveillance are practised, hence exploited in a sustainable manner. Finally, the co-existence of two stable equilibria points separated by a saddle corresponds to co-existence of two kinds of fishery; an over- exploited fishery with a large economic activity with the stock facing extinction and an under - exploited fishery with a small economic activity with the stock maintained at high levels far from extinction. Bifurcation diagrams obtained show how equilibria changes from stable to unstable at critical value of the carrying capacity as the fish population and harvesting effort vary. From the analysis, setting of appropriate fishery management practices like fishing quotas, shall guide sustainable exploitation of the Kenya commercial marine fish species. This shall be enhanced by more monitoring on fish stock density and total catches.

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List of variables and Parameters

n(t)	:	Fish population
E(t)	:	Harvesting Effort
p(t)	:	Market price
r	:	Intrinsic growth rate
k	:	Carrying capacity
T	:	Threshold population

LIST OF ABBREVIATIONS

MSY : Maximum Sustainable Yield

EEZ : Exclusive Economic Zones

YFT : Yellow Fin Tuna

IUCN : International Union for the Conservation of Nature

MDGs : Millenium Development Goals

SDGs : Sustainable Development Goals

CHAPTER ONE

INTRODUCTION

1.1 Introduction

This chapter provides a background information to the study and the description of the procedures to be used. The rest of the chapter is arranged as follows: Section 1.3 provides a description of aggregation method. Assumptions made in this study are in Section 1.4. The statement of the problem and its objectives are presented in Sections 1.5 and 1.6 respectively. Section 1.7 presents methods of study in the research. Significance of study is presented in Section 1.8.

1.2 Background of the study

In this sub section, we provide an over- view on exploitation of renewable resources, mathematical fishery modeling, Allee effect and the national outlook of commercial marine fish species in the Kenyan coastal waters.

1.2.1 Exploitation of Renewable resources

Resources are said to be renewable if they posses the capacity to self regenerate, enhancing commercial harvesting of the resources by humans in perpetuity. More Often, an assumption is made that harvesting of fish do not have a direct threat to the fish stock collapse as fish in marine waters are generally very fertile and the ocean coverage is expansive. However, humans often over-harvest fish species to near extinction or actual species collapse, Bridson [34].

In Bio-economics, exploitation of renewable resources has been considered from control theory perspective Barpan[11], Clark[14] focusing on the existence of a maximum sustainable yield (**MSY**), this is the maximum fraction that can be removed from the stock in a time period without causing the population to decline below the optimum level. This theory however, is not essentially the most appropriate management method Kar T.K and Matsuda H. [19], since the long run consumption profile does not coincide with that of utility maximization. The renewable resource stock exploited under the maximum sustainable yield is not optimum with respect to production due to the positive relationships between productivity in harvesting activities and the resource stock size. MSY policy only considers the returns of resource exploitation but totally disregards the cost operation of exploiting the resource.

1.2.2 Fishery modeling

Many fishery models, consider two state variables, $n(t)$; the fish resource with mass in kg per unit area and $E(t)$; the harvesting effort. Earlier contributions had models which read as

$$\begin{aligned} \dot{n}(t) &= f(n) - h(n,E) \\ \dot{E}(t) &= \beta(ph(n,E) - cE), \end{aligned} \tag{1.1}$$

where $f(n)$ and $h(n,E)$ are the growth and harvesting functions respectively, p the price of a unit fish resource, c, the cost per unit of harvesting the resource and β an adjustment parameter. Smith [32] considered time continuous version while Barbier et al. [7] considered time discrete version. These mathematical models have two equations; the first equation shows the evolution with time of the fish stock density determined by the natural growth and fishing. The other equation describes the variation with time of the number of vessels involved in resource exploitation and is assumed to increase when the fishing is economically viable and conversely. Reference is also made to the fishery models in Mchich *et al.*[28], where the fishery was assumed to have logistic growth with $f(n)$ taking the form

$$f(n) = rn(1 - \frac{n}{k}),$$

with r the intrinsic growth rate and k, the carrying capacity in the first Equation of (1.1) and harvested according to Type 1 functional response commonly known as the Schaefer function where

$$h(n, E) = qnE,$$

with q, a positive constant called capturability.
Several models involving Allee effect phenomena in a single equation model or a predator-prey system have also been considered, see for instance Wentworth C. *et al.*[13], Elaydi[15], Kar T.K and Matsuda H.[19] with many considering exploitation basing on the Maximum sustainable yield policy with a particular interest on how it depends on the initial population.

1.2.3 Allee effect

Over-fishing can cause the population of a fishery to persist at low densities. For centuries, North Atlantic Cod fishing was a main economic activity in the Newfoundland in Canada Bridson[34], [16]. The area was settled, in fact due to its vast stock of cod and other Atlantic fish. However, in 1992 cod population had declined so drastically that the government of Canada placed a total ban on cod fishing in the area. Data then suggested that as much as sixty percent of the adult Cod species had been harvested in a row for several years. The drastic drop in Cod populations was a major setback to the region's economy. Consequently, for months in 1996, the unemployment in the Burin Peninsula in Newfoundland was highest in Canada. The North Atlantic Cod, that had been thought to be an inexhaustible resource had not been inexhaustible at all.

Nile perch, *Lates Niloticus*, a fresh water fish was introduced in Lake Victoria in 1954 by the British government. In the period between 1980-1990, before the introduction of fish processing plants, Nile perch was favored by ordinary families that could not afford more expensive Tilapia, see Azeroul *et al.*[6]. Between 1992-2004, Nile perch became a delicacy of the elite European countries due to valuable omega-3 fatty acids in the fish that help to check heart problems and high blood pressure. Nile perch stocks have steadily dwindled with increased vulnerability of the stock reported with a drastic decline in stock abundance from 1.44 million tonne in 2006 to 0.55 million tonne in 2008. Though a gradual increase was reported between 2009 and 2011 up to 1.23 million tonne in 2014 but with dominance of juvenile fish. This further depletion compromised the reproductive capacity of the stock hence a risk of stock collapse.

The above scenarios can be attributed to a phenomenon known as Allee effect in the study of population, which is associated to the biologist Warder C. Allee [1],[12]. Allee suggested that the per capita birth rate declines when the population is at low densities. Under such circumstances, a population may persist at that low density or eventually become extinct. Stephens, Sutherland and Freckleton [33] defined Allee effect as " a positive relationships between component of individual fitness and either numbers or density of conspecific". In classical study of population dynamics, we have a negative density dependence, that an increase in density causes a decrease in fitness. Allee effect however, postulates an increase in density causes fitness of individuals to increase.

1.2.4 Outlook of Kenyan Commercial Marine fisheries

Kenya, lies within the rich tuna belt of west Indian ocean where twenty five percent of the world's tuna is caught. The tuna family comprises of Kawakawa, *Euthynnus affinis*, Skipjack tuna, Bigeye tuna, *thunnus obesus* and the dominant Yellow fin tuna, *thunnus albacares.* Yellow fin tuna is an open pelagic and oceanic species which schools primarily by size either in monospecific or multispecies groups. It is estimated at between 150 000 to 300 000 metric tonnes per year, where the current harvested stock from this is nearly 9 000 metric tonnes in a year. At the moment, Kenya has one factory installed that has a processing capacity of 105 metric tonnes per day for tuna hence accounts for less than five percent of the western Indian ocean's processing capacity. The tuna development and management strategy [24], targets to change tuna fisheries into productive and sustainable modern commercial based coastal and oceanic fisheries with immediate impacts on employment, wealth creation, improved economic outcomes and foreign exchange earning. The strategy was scheduled to run from 2013 - 2018 aimed at growing Kenya's largely undeveloped tuna supply chain that has artisanal harvesting vessels not capable of going beyond 20nm to undertake tuna harvesting to be transitioned to modern commercial fisheries in the high seas.

This strategy aligns itself with sustainable development goals **SDGs** since, according to **MDGs** report [25],[26], the first goal was to halve, between 1990-2015 the proportion of people who suffer from hunger. Hunger is still a global challenge and undernourished people globally stand at 870 million which is an eighth of the World's population where Kenya is not an exception. Provision of food and nutrition to the populace must be given a priority in attaining the rest of the seven millennium development goals. Fishery provides cheap animal protein due to availability of large water bodies. United Nations report indicates that 200 million people in the world are depended on fishing as their food source or livelihood. Kenya, is faced with the challenge of food security and malnutrition among the citizens, sustainable exploitation of fishery resources is essential to enhance the economic benefits associated with its exploitation and at the same time ensuring that the fish population do not suffer extinction. Thus, sustainable tuna resource exploitation is inevitable to avoid the Newfoundland Cod scenario since Yellow fin tuna has registered a 33 percent decline globally and now declared as a threatened species , see for instance Azeroul *et al.*[6], IUCN [18], Report [23].

1.3 Aggregation

Aggregation refers to the process in which the dimension of a dynamical system is reduced for ease of mathematical analysis, see Poggiale *et al.*[29]. Aggregation can be attained either through perfect aggregation or approximate aggregation.

1.3.1 Perfect Aggregation

This is the exact replacement of a large scale mathematical model system that involves a huge number of variables with a condensed aggregated version. The condensed system is written in a form and combination of new global variables which are defined by the first integrals of the large scale system that are assumed to be conservative. For example, consider a case of prey and predator populations with densities n_1, n_2, and n_3 respectively, such that

$$\begin{aligned} \dot{n}_1 &= r_1 n_1 (1 - \frac{a_{13} n_3}{k_1}), \\ \dot{n}_2 &= r_2 n_2 (1 - \frac{a_{23} n_3}{k_2}), \\ \dot{n}_3 &= -r_3 n_3 + (a_{31} n_1 + a_{32} n_2) n_3, \end{aligned} \tag{1.2}$$

where r_i, for $i = 1, 2, 3$, a_{13}, a_{23}, a_{31} and a_{32} are parameters is a system of three equations that can be aggregated to a new system with equations in variables $Y_1 = n_1 + n_2$ and $Y_2 = n_3$, where the prey is aggregated into a single compartment and the parameters in Equation (1.2) satisfy the following set of relations:

$$\begin{aligned} r_1 &= r_2 = r, \\ k_1 &= k_2 = k, \\ a_{23} &= a_{13} = a, \\ a_{32} &= a_{31} = a'. \end{aligned} \tag{1.3}$$

The aggregated system now becomes,

$$\begin{aligned} \dot{Y_1} &= rY_1(1 - \frac{aY_2}{k}), \\ \dot{Y_2} &= -r_3Y_2 + a'Y_1Y_2. \end{aligned} \tag{1.4}$$

These set of relations are quite restricted and take particular values of the parameters.

When perfect aggregation is possible, the aggregated model is a condensed prototype of the initial model. In practice, perfect aggregation is sometimes impossible to realize since the conditions for condensing the large system are restrictive and may only be possible where the parameters of the large system satisfy very specific relationships, Poggiale *et al.*[29].

1.3.2 Approximate Aggregation

This corresponds to the replacement of a large scale mathematical model by an aggregated system, obtained by appropriate approximation. The approximation is realized by justified simplification. This is easily attained when some variables are on a fast time scale with respect to others. Variables on a fast time scale rapidly attain equilibrium and the approximation involves replacement of the fast variables with non trivial equilibrium values. For Instance, incase fast variables approach an asymptotically stable point, they are replaced with non trivial equilibrium values, while if the variables on a fast time scale are periodic functions of time, they are replaced with their time averages. In other cases, approximation is done by neglecting the evolution of very slow variables which do not vary during the period of observation of the system. This is the type of aggregation used in this study.

1.4 Assumptions of the study

In this study the following assumptions have been made:

(i) Effects of adverse environmental variations and diseases have little impact on the Fishery;

(ii) The harvesting of the fish species is a continuous process;

(iii) The variation of the market price of the fish is more rapid than the fishing effort and the growth of the fish stock.

1.5 Statement of the problem

Fishery models have mainly considered logistic equation for its population growth either continuous or discrete with type1 functional response for resource exploitation, see for instance Arino[3], Auger *et al.*[4], Barbier[7]and Mchich *et al.*[28]. Fish, exhibits co-operative and social characteristics among individuals which increases its survival through predator surveillance and enhanced spawning. Individual reproduction rates are known to decrease if the density is below particular critical values due to difficulty in finding mates. A population growth equation, with a depensation term, to address Allee effect phenomena induced by over-exploitation is thus more suitable to investigate the dynamics of a fishery. The threshold characteristics determines whether the population will collapse and not recover in foreseeable time, which is of great concern to conservationists and economists interested in sustainable fishery resource exploitation. This research presents analysis of a mathematical fishery model that takes into account three main aspects; growth of the fish population with Allee effect, harvesting effort and the market price of the fish resource. The results form a basis to guide sustainable exploitation of commercial fish stock in the Kenya coastal marine waters.

1.6 Main objective

To develop and study the dynamics of a fishery model under exploitation, with Allee effects in the population.

1.6.1 Specific objectives

The specific objectives of this research are;

(i) To Formulate a mathematical fishery model with over-fishing induced Allee effects in its population growth.

(ii) To analyze the long term solutions of the model in (i)

(iii) To Investigate and identify the threshold population value T for sustainable fishery exploitation with Kenyan coastal marine commercial fish species as an example.

1.7 Methods of Study

The following methods are used in this research study;

(i) Model formulation.

We formulate a model using a set of ordinary differential equations. The equations in this study are formulated with population, harvesting effort and the market price as dependent variables whereas the independent variable is time.

(ii) Aggregation.

This is the reduction of the dimension of a dynamical system for ease of analysis. It is attained through either perfect aggregation or approximate aggregation. We employ approximate aggregation in this study.

(iii) Numerical simulation.

MATLAB software is used to generate numerical simulations to determine the long term behavior of the fishery model.

1.8 Significance of the Study

Optimal exploitation of a fishery resource is a concern for economists whilst species extinction is a stark reality for conservationists. This research sought to identify the threshold population value for which sustainable fishery resource exploitation can be done and suggest fishery management practices to curb eminent species extinction. This is with respect to the Kenya Tuna Fisheries development and management strategy of 2013-2018, which is consistent with the aspirations of the national development blueprint, the Vision 2030 and the Kenya Oceans and Fisheries policy. In these policy documents, sustainable exploitation of under-exploited Exclusive Economic Zones(**EEZ**) is of foremost priority.

1.9 Justification of the study

The results obtained in this study forms a basis for setting relevant parameters like capturability q and threshold population T. This is subject to availability of data on stock density and total catches. These parameters will guide setting of fishing quotas which shall enhance sustainable exploitation of the commercial fish stock in the Kenya coastal marine waters.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In this chapter, we review the literature on fishery modeling dynamics. Discrete and continuous time models are discussed in section 2.2, models with delay differential equations are discussed in section 2.3 while in section 2.4, models with an Allee effect are discussed.

2.2 Discrete and continuous time models

In Smith [32], the following model is considered ;

$$\begin{aligned} \dot{n} &= f(n) - h(n, E) \\ \dot{E} &= \beta(ph(n, E) - cE). \end{aligned} \tag{2.1}$$

with $n := n(t)$ representing the mass of the fish stock resource, $E := E(t)$ representing the harvesting effort at time t. The function $f(n)$ represents the natural growth of the fish resource, $h(n, E)$ the harvesting term that depends on the fish resource and the effort of fishing. The harvesting function $h(n, E) = g(n, E)E$, where $g(n, E)$ is the amount of the fish captured per unit time and per unit of fishing effort commonly referred to, as a functional response, see Murray [27]. In this model, a Holling Type II functional response is used. The constant c is the cost of fishing effort per unit of resource, β is a positive co-efficient of adjustment that depends on the fishery and p , the price per unit landed fish stock of the fish resource at time t. Analysis of bifurcation and stability for the model in Equation (2.1) reveals two positive equilibria that are stable and a saddle. Barbier [7], studied a discrete version of (2.1) in which the functional response used was a

Schaefer function, $g(n, E) = qn$ with q, a constant always positive called capturability, Schaefer[30]. Stability and time pooled analysis of the parameters revealed existence of a saddle, and two stable fixed equilibrium point. The work do not consider market price as a variable in the fishery model.

Mchich *et al.*[28] considered the model;

$$\begin{aligned}
\grave{n}_1 &= kn_2 - k'n_1 + \epsilon\{r_1 n_1(1 - \frac{n_1}{k_1}) - q_1 n_1 E_1\}, \\
\grave{n}_2 &= k'n_1 - kn_2 + \epsilon\{r_2 n_2(1 - \frac{n_2}{k_2}) - q_2 n_2 E_2\}, \\
\grave{E}_1 &= mE_2 - m'E_1 + \epsilon E_1\{pq_1 n_1 - c_1\}, \\
\grave{E}_2 &= m'E_1 - mE_2 + \epsilon E_2\{pq_2 n_2 - c_2\},
\end{aligned} \tag{2.2}$$

where the grave indicates the differentiation with τ, a faster scale of time as independent variable in which a fishery of two patches with two fishing areas is studied. It is assumed that the fish resource as well as the fishing vessels moves rapidly between the two fishing zones. The first two equations show the growth of two fish species with population n_1 and n_2, dispatched on the two fishing patches with migration rates k' and k respectively, and exploited by two vessels. The last two equations of the model show the evolution of the harvesting efforts E_1 and E_2 by the fishing boats with fishing rates m' and m in the two zones respectively. Each equation in the model (2.2) has two parts; one describes the migration between the fishing patches at a faster time scale τ and the other part describes the natural growth and the capturability for fish species and the harvesting efforts at a slower time scale t. With the assumption of rapid movement of the fish species and harvesting vessels between the two zones, the model is a system completely at a fast time scale $\tau = \epsilon\ t$; $\epsilon << 1$ with respect to a slower time scale t. The constants r_i, k_i and q_i for $i = 1, 2$ denote the intrinsic growth rates, the carrying capacities and capturability co-efficients respectively in the i^{th} patch. Bifurcation and stability analysis of the model in Equation (2.2) revealed stable equilibrium and stable limit cycle, their existence depending on parameters k' , k ,m' and m . These results were obtained with an assumption of a fixed market price which is only relevant to a fishery with small economic activity.

Auger *et al.* [4], modified Equation (2.2) by adding an equation in market price. The dynamics of Equation (2.2) under effects of linear market price was studied. The addition of the fifth equation realised;

$$\begin{aligned} \grave{n}_1 &= kn_2 - k'n_1 + \epsilon\{r_1 n_1(1 - \frac{n_1}{k_1}) - q_1 n_1 E_1\}, \\ \grave{n}_2 &= k'n_1 - kn_2 + \epsilon\{r_2 n_2(1 - \frac{n_2}{k_2}) - q_2 n_2 E_2\}, \\ \grave{E}_1 &= mE_2 - m'E_1 + \epsilon E_1\{pq_1 n_1 - c_1\}, \\ \grave{E}_2 &= m'E_1 - mE_2 + \epsilon E_2\{pq_2 n_2 - c_2\}, \\ \grave{p} &= \alpha\{D(p) - (q_1 n_1 E_1 + q_2 n_2 E_2)\}, \end{aligned} \tag{2.3}$$

with $p = p(t)$ as the market price of a unit fish stock at time t, α constant that describes the speed at which price is adjusted on the market while $D(p)$ is a function of demand assumed to decrease when price increases. The last equation of (2.3) represents the evolution of the price with time and it is taken that variation in price is due to the difference between the demand $D(p)$ and the supply $S(p) := q_1 n_1 E_1 + q_2 n_2 E_2$, which is the catch. A further assumption that the evolution of the price of the catch is with respect to a faster time scale is made. Since two time scales are assumed, model aggregation of (2.3) is achieved by calculating the fast equilibrium by setting $\epsilon = 0$ and then assuming that faster dynamics attains stable equilibrium state. These fast equilibrium equations are substituted in model in Equation (2.3) to become;

$$
\begin{aligned}
\dot{n} &= n\{r(1-\frac{n}{k})-qE\}, \\
\dot{E} &= E\{-c+qn(A-qnE)\},
\end{aligned}
\tag{2.4}
$$

a system in two dimensions of differential equations ordinary in nature that governs the total fish stock $n(t) = n_1(t) + n_2(t)$ and the total harvesting effort $E(t) = E_1(t) + E_2(t)$ at a slower time scale t, such that

$$
\begin{aligned}
r &= r_1{v_1}^* + r_2{v_2}^* \\
\frac{r}{k} &= \frac{r_1{v_1}^{*2}}{k_1} - \frac{r_2{v_2}^{*2}}{k_2} \\
q &= q_1{v_1}^*{\mu_1}^* + q_2{v_2}^*{\mu_2}^* \\
c &= c_1{\mu_1}^* + c_2{\mu_2}^*, \\
{v_1}^* &= \frac{k}{k+k'}, \\
{v_2}^* &= \frac{k'}{k+k'}, \\
{\mu_1}^* &= \frac{m}{m+m'}, \\
{\mu_2}^* &= \frac{m'}{m+m'},
\end{aligned}
\tag{2.5}
$$

with ${\mu_1}^*$ and ${\mu_2}^*$ representing the proportion of the harvesting effort on each patch while ${v_1}^*$ and ${v_2}^*$ gives the proportion of fish resource in each zone at a faster equilibrium and c_1 and c_2 the cost per unit of harvesting effort on each patch at a faster equilibrium. When bifurcation and local stability analysis is done for the model in Equation (2.4), two stable positive equilibria are found to co-exist with a separatrix of the stable manifold in the middle . This shows that two very different stable equilibria can exist for the same fishery, one corresponding to an over-exploitation of the resource and the other, to a traditional fishery in which the fish stock is maintained at large level which is with little risks of becoming extinct but does not permit a meaningful economic exploitation. These two stable equilibria co existing causes a dilemma since each does not enhance sustainable economic fishery exploitation.

Steinworth *et al.*[8] considered a fish population model under the effects of over harvesting using the equation;

$$\dot{y} = -ry(1 - \frac{y}{T})(1 - \frac{y}{k}) - Ey \tag{2.6}$$

for $y > 0$, $0 < T < k$, where r, T and k are parameters of a particular species of fish population whose population is denoted by $y(t)$ and $E(t)$ is the harvesting effort. Built on assumptions:

(i) Rate of harvesting of the fish by the humans is the only factor which changes the equilibrium states of the fish population and it is a constant.

(ii) Factors that limit population such as diseases, climatic conditions and predation would not vary and would not be affected by harvesting.

(iii) Harvesting activity did not affect reproduction behaviour of the fish.

Model in Equation (2.6), is a modified logistic growth with resource dependent harvesting. In this model, there is a realization that there is an optimum population below which meaningful reproduction cannot be sustained hence the population becomes extinct. This is taken care of by a threshold population T such that the population grows logistically as in the model

$$\dot{y} = ry(1 - \frac{y}{k}), \tag{2.7}$$

when $y > T$ but declines and cannot recover once y is below T, see [8] for instance.

The equilibrium solution of (2.6) yields two values of the population;

$$y_k = \frac{kT[r(\frac{1}{k} + \frac{1}{T}) + \sqrt{r^2(\frac{1}{k} + \frac{1}{T})^2 - \frac{4r(r+E)}{kT}}}{2r} \tag{2.8}$$

and

$$y_T = \frac{kT[r(\frac{1}{k} + \frac{1}{T}) - \sqrt{r^2(\frac{1}{k} + \frac{1}{T})^2 - \frac{4r(r+E)}{kT}}}{2r} \tag{2.9}$$

the limiting population and the threshold population respectively both dependent on the fishing effort. Since y_k and y_T depends on E, sustainable rate of fishing is determined for each particular E as

$$y_E = y_k E.$$

Using the raw data of the Chinook Salmon fish population, model parameters were examined and analyzed determining the sustainable fishing rate and the absolute fishing rate. The model in Equation (2.6) makes assumption that the harvesting is constant whereas in commercial fisheries, harvesting is a variable depending on the difference between the returns and harvesting expenses. Moreover, the assumption that fish stock reproduction behavior was not changed by fishing activity is unrealistic since fish exhibits social characteristics which are affected by harvesting.

2.3 Models with delay differential equations

With a logistic equation

$$f(n) = rn(1 - \frac{n}{k}), \tag{2.10}$$

P. Auger and Arnould Ducrot [5] considered a fishery model with fish stock involving a delay given by;

$$\begin{aligned} \dot{n} &= rn(1-\eta) - \rho(n, E), \\ \dot{E} &= P[(1-\eta)\rho(n, E) + \delta S)] - cE, \end{aligned}$$

$$\dot{S} \;=\; \eta\rho(n, E) - \delta S. \tag{2.11}$$

The variables and parameters in Equation (2.11) have the same meaning as in Equation (2.1) except for S, a new stock variable in which it was assumed that some fraction of the harvested fish is immediately sold while the remaining is stocked for processing and later enters the market. The fraction $\eta \in [0, 1]$ of the fish harvest enters to the stock compartment, the remaining fraction $1 - \eta$ is sold immediately to contribute to the harvesting effort E. The parameter $\delta > 0$ denotes the rate at which the fish in the stock compartment is returned to the market. By integrating the third equation and eliminating S, the system in Equation (2.11) is reduced to a two component delay differential equations:

$$\begin{aligned} \dot{n} &= rn(1-n) - \rho(n, E), \\ \dot{E} &= (1-\eta)pqnE + \eta pq \int_{-\infty}^{0} w(\theta)n_t(\theta)E_t(\theta)d\theta - cE, \end{aligned} \tag{2.12}$$

with $n_t(\theta) = n(t+\theta)$ and $E_t(\theta) = E(t+\theta)$ as historical functions while $w(\theta) = \delta e^{\delta\theta}$ is a weighted function taking a simple exponential damped rate with $\theta \in (-\infty, 0)$. Stability and bifurcation analysis for Equation (2.12) shows that for each continuous, bounded initial condition (n_0, E_0), the model has a defined global solution $(n(t), E(t))$ for $0 \le n(t) \le 1$, $E(t) \ge 0$ $\forall t > 0$. The model, considers a fixed market price whereas it is known that the price of resources varies as a determined by the difference between demand and supply. Commercialized fish species risks extinction and the use of the logistic growth function in a such an over-exploited species do not address threats of extinction.

Arino O. *et al.* [3], considered the delayed logistic model (Hutchinson's equation) for population growth

$$\dot{n} = rn(t)[1 - \frac{n(t-\tau)}{k}], \tag{2.13}$$

where r and k are as defined in the logistic equation and $\tau > 0$ is a constant. With initial value as

$n(\theta) = \phi(\theta) > 0$, $\theta \in [-\tau, 0]$

with ϕ continuous on $[-\tau, 0]$, Equation (2.13) has equilibria $n = 0$ which is unstable and a positive equilibrium $n = k$. Stability analysis of the positive equilibria yields different results of stability; for $0 \leq r\tau \leq \frac{\pi}{2}$ the equilibrium point is positive and asymptotically stable, for $r\tau > \frac{\pi}{2}$ the positive equilibrium is unstable and when $r\tau = \frac{\pi}{2}$, Hopf bifurcation occurs at $n = k$,with bifurcation of periodic solutions from $n = k$. These periodic stable solutions exist for $r\tau > \frac{\pi}{2}$.

Gopalsamy [17], studied a single population species model with Allee effects. In the model, the per capita growth rate is a density dependent quadratic function subjected to time delays given by;

$$\dot{x} = x(t)[a + bx(t - \tau) - cx^2(t - \tau)] \tag{2.14}$$

where $a > 0$, $c > 0$,$\tau \geq 0$ and b are positive constants. In the single equation model, when the population density is high, the compensatory effects of aggregation and social cooperation are dominated by density dependent stabilizing negative feedback effects caused by intraspecific competition. It was found that the positive equilibrium point of Equation(2.14),

$$x^* = \frac{b \pm \sqrt{b^2 + 4ac}}{2c} \tag{2.15}$$

is conditionally globally attractive. But if the delay is sufficiently large, solutions of Equation (2.14) oscillate about the positive equilibrium and if $\tau x^*(2cx^* - b) \leq \frac{3}{2}$, then the positive equilibrium x^* attracts all positive solutions. The model equations in Gopalsamy [17] and Arino[3] do not consider fishing mortality hence not suitable for studying fishery population dynamics.

2.4 Models With Allee Effect

Wentworth C. *et al.*[13], constructed a harvest strategy of a fish population with Allee effect using the model

$$\dot{n} = n(n-a)(1-n) - fn, \tag{2.16}$$

where a denote the Allee effect such that $0 < a < 1$, $n(t)$ the fish population at time t and f the fishing rate. The equation (2.16) has two equilibrium population density given by

$$n^* = \frac{-1 - a \pm \sqrt{a^2 + 6a + 4f^* + 1}}{2}.$$

Using implicit differentiation and optimization, two solutions for the maximum yield are obtained as;

$$Y_1^* = -\frac{1}{18}[1 + a(8+a) - \sqrt{(1+a)^2(1+5a+a^2}] \times [-1 - a + \sqrt{1 + 6a + a^2 + 4f},$$

which is unstable and

$$Y_2^* = \frac{1}{18}[1 + a(8+a) + \sqrt{(1+a)^2(1+5a+a^2}] \times [-1 - a + \sqrt{1 + 6a + a^2 + 4f}$$

which is stable. The fish yield over time γ given by

$$Y = \int_0^\gamma fn(t,f)dt,$$

the yield was maximized by taking

$$\frac{d}{df}Y = [n(t,f) + f\partial_f n(t,f)]dt = 0.$$

An algorithm for calculating the optimum harvest rate was determined and similarly, for various values of γ, sustainable population density solutions with harvest rates that maximize yield are determined, dependent on the initial population. The results only focus on the **MSY** and considers only the population dynamics in the presence of fishing without other factors that affect fishing effort like the market price of the harvested fish.

Kar T.K. and Matsuda H. [19], considered a predator - prey system with an Allee effect in the growth of the prey(fish) given by

$$\begin{aligned} \dot{x} &= rx(1-\frac{x}{k})(\frac{x}{L}-1)-mxy-qEx, \\ \dot{y} &= m\alpha xy-dy, \end{aligned} \tag{2.17}$$

where m and α are the interaction parameters between the predator (y) and the prey (x), d is the death rate of the predator while L is the threshold population in the prey. This was premised on the fact that marine ecosystem are interlinked and competition and predator - prey are the common interaction for fishery. The system in (2.17) has interior equilibrium (x^*, y^*), with

$$x^* = \frac{d}{m\alpha}$$

and

$$y^* = \frac{1}{m}[r(1-\frac{d}{m\alpha k})(\frac{d}{m\alpha L}-1)],$$

which is locally asymptotically stable if

$$x^* > (\frac{k+L)}{2}$$

and unstable if

$$\frac{d}{m\alpha} < (\frac{k+L)}{2}.$$

Using the optimal harvesting problem

$$J = \int_0^\infty e^{-\delta t}(pqx - (c+\tau)Edt,$$

bang - bang control is applied to the system in Equation (2.17) from the initial state (x_0, y_0) and the optimal policy for harvesting obtained with the optimal point being globally asymptotically stable. This model does not consider market price as a variable which is more relevant to the commercial fishery.

Saber N. and Robert J. Sacker [15], developed several models of Allee effect using discrete maps and made a modification to the classical Ricker map to control the extinction of the population and enhance the return of the population to the carrying capacity in the concave Beverton-Holt maps. These single equation models do not consider other factors like harvesting effort and market price, hence not suitable for the study of commercial fishery dynamics.

Although mathematical fishery models exist for both ordinary differential equations, delayed differential equations and difference equations, an attempt made to model dynamics of a fishery with population growth exhibiting Allee effect mainly considers exploitation of the fishery basing on the **MSY** theory and does not consider harvesting effort and the market price as dynamic variables. In this research, we consider a model comprised of three ordinary differential equations, with Allee effect in the population growth equation. A threshold population below which the fish population is not viable due to vulnerability to diseases, predators and inadequate reproduction is considered. Thus, we have population growth of a fish stock exhibiting Allee effect, the harvesting effort and the price changes on the market which dictates investment in the harvesting effort, as variables in this study.

CHAPTER THREE

FISHERY MODEL WITH ALLEE EFFECT

3.1 Introduction

In this chapter, we develop a mathematical fishery model with the fish population, harvesting effort and market price as variables. The model is considered as a dynamical system in which the fish population growth equation has a sparsity term to address Allee effect. General Model derivation is presented in section 3.2 while discussion on model equations is presented in section 3.3

3.2 General Model Formulation

In this study, a system of ordinary differential equations that describes the relationship between three dependent variables: the fish population $n = n(t)$, fishing effort $E = E(t)$ and market price $p = p(t)$ at time t is considered. The rate of change in population is taken to be the difference between population growth function with Allee effects and the harvesting function. The rate of change in harvesting is taken to be proportional to the difference between cost of harvesting and the total gains while the rate of change in market price is taken to be proportional to price dynamics and the difference between the demand and supply of the fish stock

A general fishery model that can be used to study the relationship between these variables is thus,

$$
\begin{aligned}
\dot{n} &= f(n) - h(n, E), \\
\dot{E} &= \beta(ph(n, E) - cE), \qquad (3.1)
\end{aligned}
$$

$$\dot{p} = \alpha p(D(p) - h(n, E)),$$

which is a time continuous model. The constants β and α are taken as positive. The function $f(n)$ is a function for growth of the fish population with the population exhibiting Allee effects that shall be accounted for by the sparsity term. The harvesting function $h(n, E)$, depends on the fish resource and the fishing effort E. Constant c is the cost per unit of the harvesting effort, p the market price per unit of the harvested fish while $D(p)$ is the demand function.

3.3 Model Formulation

In this section we detail how the equations in the market price, harvesting effort and fish population are obtained.

3.3.1 Equation for Fish Population growth With Allee effects

The first equation in (3.1), describes the rate of change in the fish resource population which is harvested. Given a fish population $n(t)$ at time t not being harvested, if b and d denote the birth rate and death rate respectively, then

$$f(n) = \dot{n} = bn - dn = rn, \tag{3.2}$$

where $r = b - d$ is the intrinsic growth rate. The analytic solution of (3.2) with the initial population $n(0) = n_0$ is

$$n(t) = n_0 e^{rt}. \tag{3.3}$$

The function (3.3) is a traditional exponential growth if $r > 0$ or a decay if $r < 0$ see Figure 3.1. Such a growth in population is only valid within a short period of

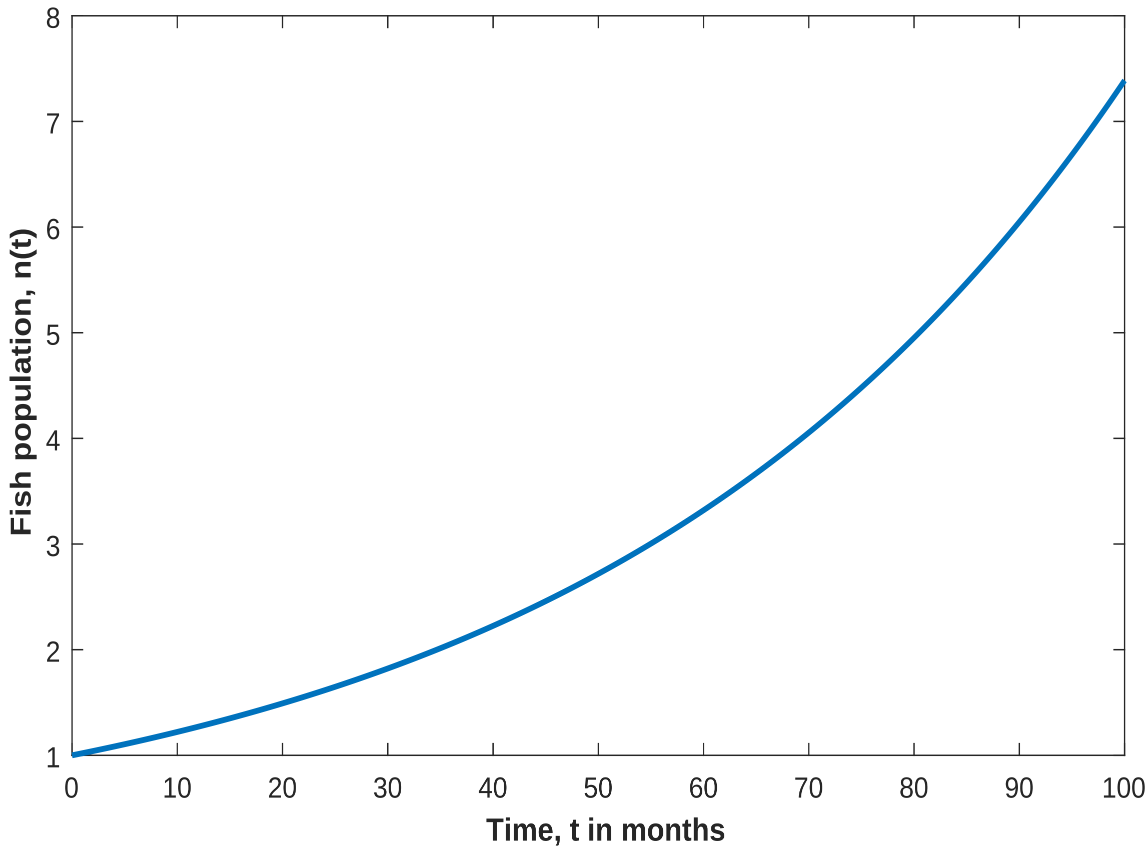

Figure 3.1: Exponential growth

time and cannot persist forever, due to population limiting factors like food supply and oxygen levels in the fishery. Taking into account that resources are limited in nature, a growth equation of the form

$$f(n) = \dot{n} = rn(1 - \frac{n}{k}), \tag{3.4}$$

has been used in several population models, see for instance Auger P., Arnould D.[5],Auger *et al.*[4] and Mchich *et al.* [28] and is referred to as the logistic growth equation. In this equation, $r > 0$ is the intrinsic growth rate and $k > 0$ is the carrying capacity of the fishery which essentially is, the quality of the fishery in terms of food availability and oxygen levels. When $n < k$, $\dot{n} > 0$ and the population grows and if the term $\frac{n}{k}$ is very small, the differential equation (3.4) is not different from (3.2). When $n > k$, $\dot{n} < 0$ and the population declines since individuals of the species compete with each other for resources that are limited. The differential equation (3.4) has two steady states; the non-trivial $n = k$, and $n = 0$. Any population above or below k will result in a growth rate that restores the population back to k, whereas for any initial population $n_0 = k$, the population remains constant. The analytic solution of (3.4) is of the form

$$n(t) = \frac{n_0 k}{n_0 + (k - n_0)e^{-rt}}, \tag{3.5}$$

see for instance Boyce W. E., Diprima C. R [10] numerically shown in Figure 3.2,

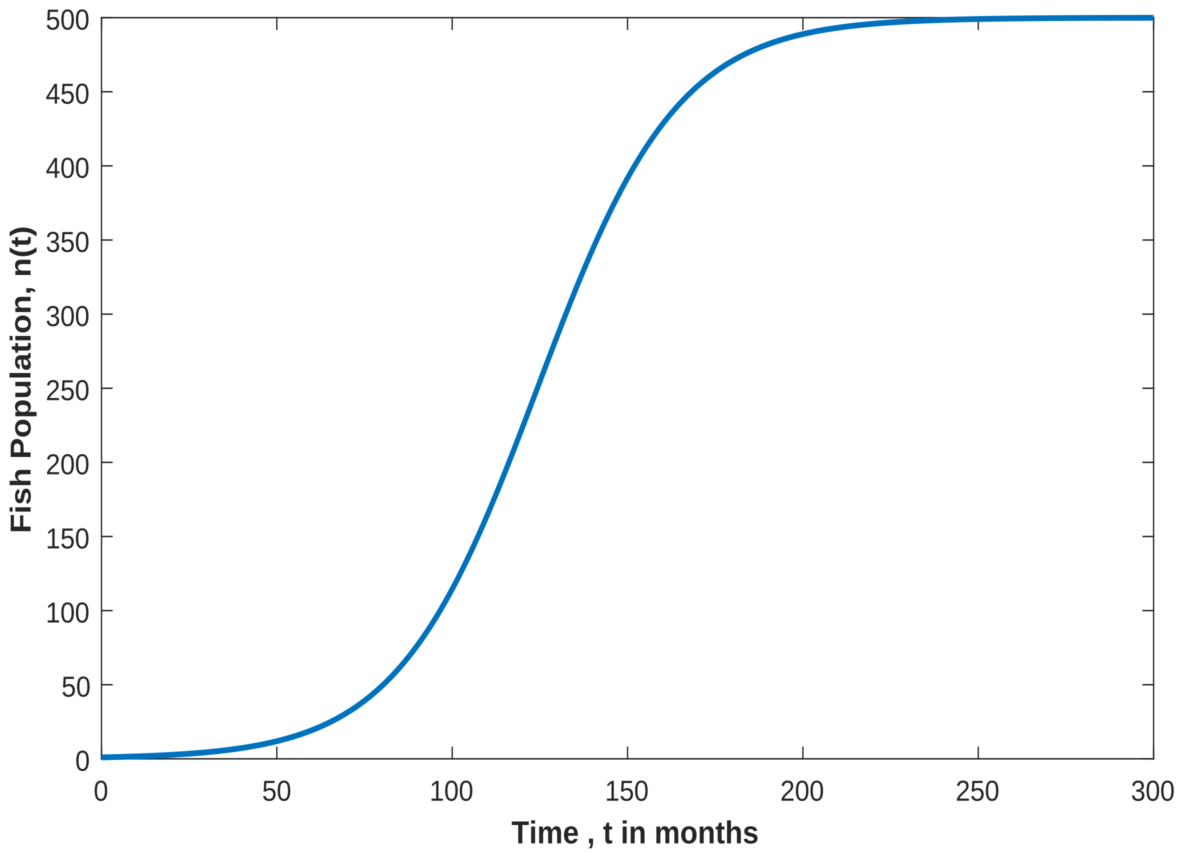

Figure 3.2: Logistic growth

It can be seen that for $n_0 < k$, there is asymptotic growth in population approaching k as $t \to \infty$, this is due to social-cooperativeness behavior in the fishery. If $n_0 > k$, the population decreases again approaching k asymptotically as $t \to \infty$. If $n_0 = k$, the population remains at $n = k$ in time. This implies that the positive equilibrium $n = k$ of (3.4) is globally stable, that is for any solution $n(t)$ of (3.4) with initial value $n(0) = n_0$, $\lim_{t\to\infty} n(t) = k$.

The assumption that the rate of growth at any time dependent is on relative number of members of the species at that time and $\dot{n} > 0$ even when n is near zero is not realistic. Moreover, the view in classical population dynamics that competition of resources causes a population to experience a reduced overall growth rate at higher density and an increase in the rate of growth at lower density may apply in certain populations but not all like the fish population which exhibits co-operative and social characteristics. In such populations, there has to be a threshold population below which reproduction cannot be effective and the population may decline irrecoverably due to reduced likelihood of finding mates, juvenile mortality and reduced predator surveillance. This phenomena in which there is a positive correlation between the population density and the fitness of individuals negating the classical view of population dynamics is referred to as the Allee effect. There are a number of equations which describes this population growth phenomena, If we consider the case in which a threshold population T is included, f(n) takes the form

$$f(n) = \dot{n} = rn(\frac{n}{T} - 1)(1 - \frac{n}{k}), \tag{3.6}$$

for $n > 0$, $0 < T < k$. The term

$$(\frac{n}{T} - 1),$$

is the sparsity term which takes account of Allee effect. The solutions of (3.6) are such that any initial population above k and T will approach k in time while initial population below T will approach zero in time. Figure 3.3 shows the solution curves of the Equation (3.6) for various initial populations with $r = 0.05$, $T = 100$ and $k = 500$.

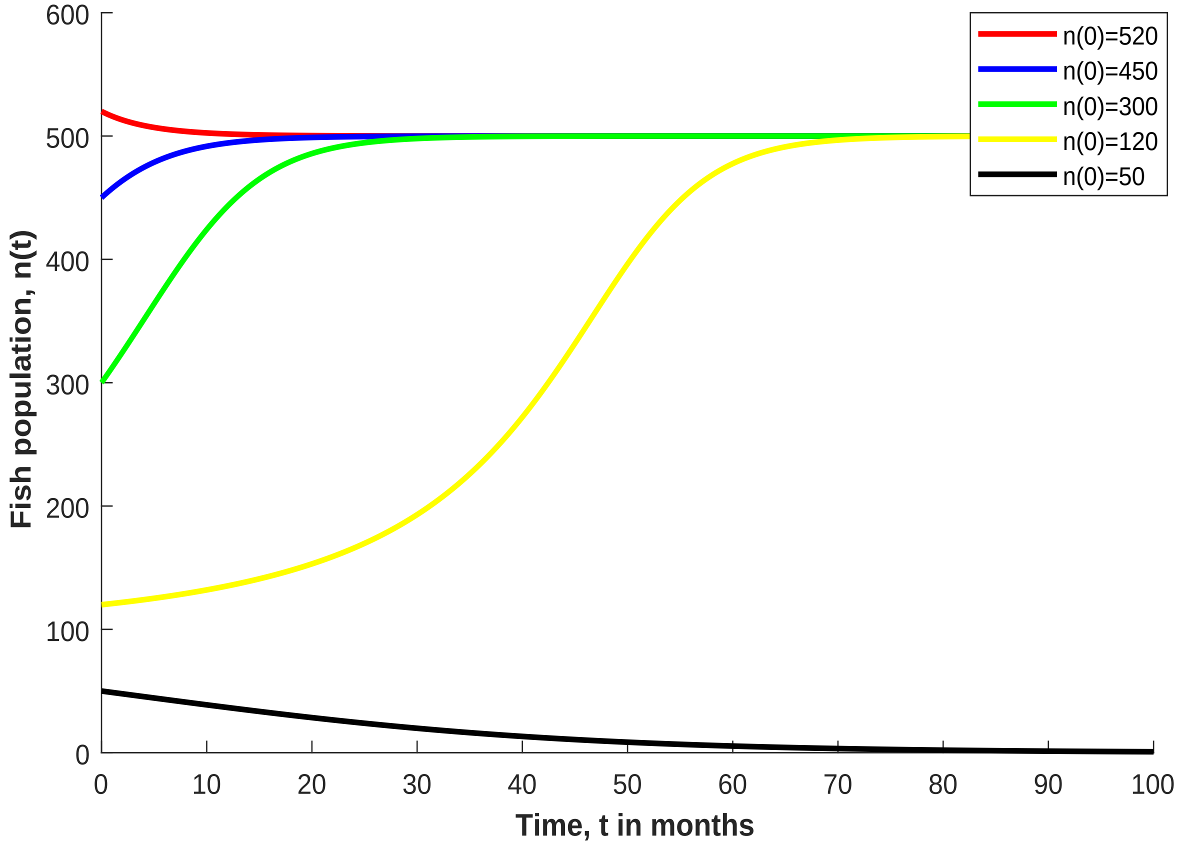

Figure 3.3: Solution curves for Fish Population Equation with zero harvesting

The population just behaves as described in Equation (3.4) when $n > T$ but declines to zero when $n < T$. If the population is above the threshold value T, there is compensation due to positive effects of density dependence associated with co-operativeness manifests making the population to grow logistically but controlled by negative density feedback making the growth approach the carrying capacity. Below the critical value, compensation weakens due to sparsity causing lower reproduction chances and juvenile mortality, the population declines approaching zero. Thus, in the first equation of (3.1) the function $f(n)$ takes the form as expressed in (3.6) where $0 < T < k$ is the critical population abundance and $k > 0$ is the carrying capacity of the fishery.

In classical predator prey models, predation mimicked by fishing, in this model it is a function written as $h(n, E) = g(n, E)E$. The function $g(n, E)$ is referred to as a functional response, the amount of the resource captured per unit of fishing effort. In this fishery model, the Schaefer function is assumed thus $g(n, E) = qn$ with q as a positive constant called capturability. This functional response depends linearly on the resource, referred to in ecological modeling as the law of mass action. This is suitable in fishery since it accounts for fishing effort and mortality as variables controlling the growth of the fish population, see Schaefer[30]. Thus $h(n, E) = qnE$. The given functions of $f(n)$ and $h(n, E)$ makes the first equation in the Model equation (3.1) becomes;

$$\dot{n} = rn(\frac{n}{T} - 1)(1 - \frac{n}{k}) - qnE \tag{3.7}$$

Numerical simulation of Equation (3.7) for $q = 1$ and a constant harvesting effort E shows that initial population below and slightly above the threshold population $T = 100$ now decays to zero, see Figure 3.4.

This implies that harvesting increases sparsity which further compromises social - cooperativeness benefits hence increasing the extinction of the species. Moreover,

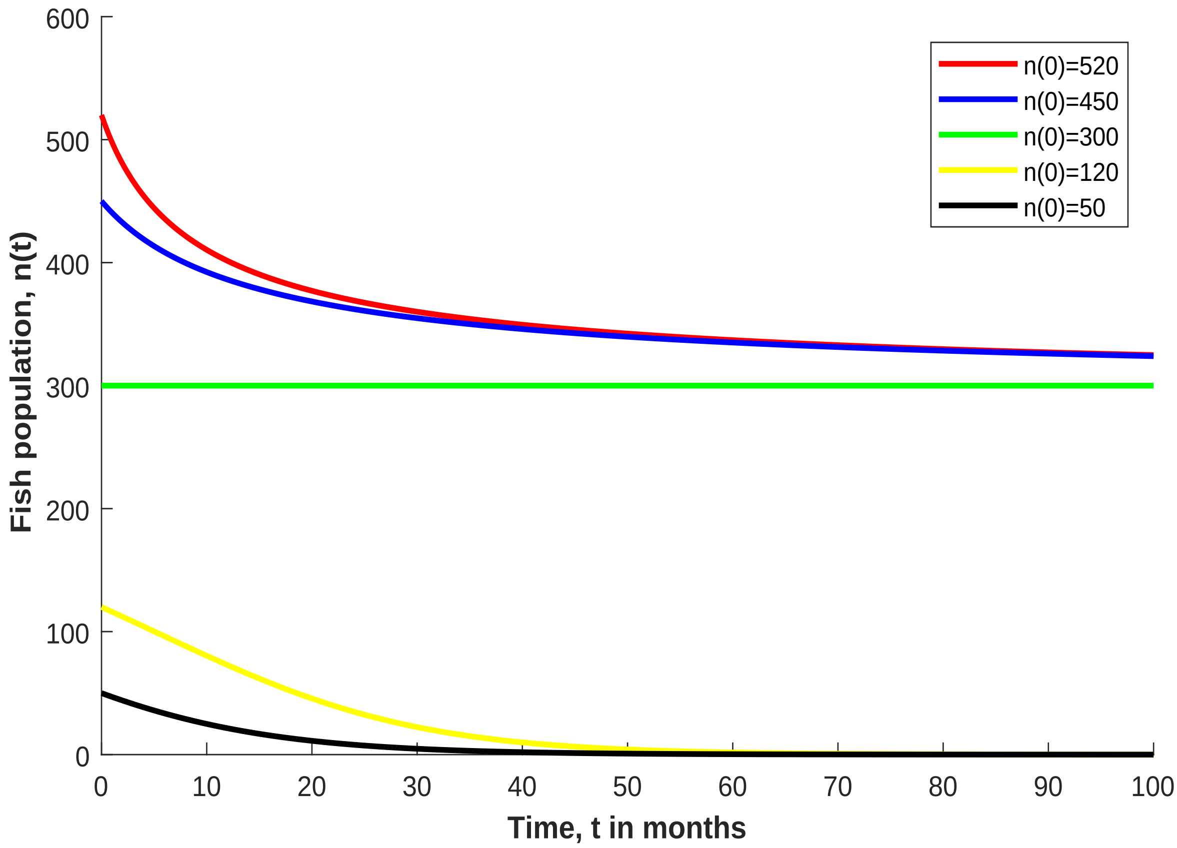

Figure 3.4: Solution curves for Fish Population Equation with minimal harvesting

initial populations below the carrying capacity grows logistically towards a lower new value of the carrying capacity $k = 350$. This is as a result of the reduced number of individuals responsible for spawning and predator surveillance. This makes the maximum population of the species that can be supported by the fishery to reduce.

3.3.2 Harvesting Effort Equation

The second equation of (3.1) describes how the fishing effort evolves depending on the difference between the total benefits and the cost incurred during harvesting.

A suitable equation that can describe the dynamics in harvesting effort is

$$\dot{E} \propto (benefit - cost),$$

with the total benefit given by the product of the market price p with the total catch $h(n, E)$. The total cost is the product of the cost per unit of harvesting c and the harvesting effort E. The equation

$$\dot{E} = \beta(ph(n, E) - cE), \tag{3.8}$$

is obtained with β a constant of proportionality which is a positive adjustment co-efficient. With $h(n, E)$ given, Equation (3.8) becomes ;

$$\dot{E} = \beta E(pqn - c). \tag{3.9}$$

3.3.3 Market Price Equation

The third equation describes how the market price p varies dependent on the demand, the supply of fish resource and the price dynamics. Relative variations in the market price are assumed to be governed by a simple balance between the supply of fish stock, the catch and its demand on the market. We represent this relation by

$$\dot{p} = \alpha p(D(p) - S(p)),$$

where $D(p)$ and $S(p)$ are functions of demand and supply of the fish stock respectively, see for instance Mackey [21]. It is assumed that the maximum supply always exceeds minimum demand, that is,

$$min_p D(p) \leq max_p S(p).$$

The argument for demand is such that for any change in demand, a simplest assumption that consumers base all buying decisions on the current market price $p(t)$ is taken. Thus, a demand function linearly dependent on the market price p, is chosen as

$$D(p) = A - p(t),$$

with A a positive constant parameter that represents the limit threshold of the market price, see for instance Arne E. *et al.* [2], Lafrance [20]. In this case, there is a linear decrease in demand as price increases . This is suitable for resources whose price is sensitive with a unit marginal cost since their consumption is dependent on the availability of their substitutes. The argument of the supply schedule for the stock is made basing on its dependence on the captured fish stock. Therefore,

$$S(p) = qnE. \tag{3.10}$$

These functions of supply and demand make the Equation in market price to become ;

$$\dot{p} = \alpha p(A - p - qnE). \tag{3.11}$$

Equation (3.11), has nonlinear variation in market price depending on the difference between supply and demand, see for instance Makwata H. *et al.*[22].

Equations (3.7),(3.9) and (3.11) constitute the complete time continuous model which shows the evolution of the three variables given by,

$$\begin{aligned}
\dot{n} &= rn(\frac{n}{T} - 1)(1 - \frac{n}{k}) - qnE, \\
\dot{E} &= \beta E(pqn - c), \\
\dot{p} &= \alpha p(A - p - qnE),
\end{aligned} \tag{3.12}$$

CHAPTER FOUR

THE MODEL ANALYSIS AND RESULTS

4.1 Introduction

In this chapter, we aggregate equations in (3.12) to reduce them to a two dimensional system, obtain its equilibrium points and analyse their local stability in order to study the long-term behaviour of the solutions of the system. Bifurcation analysis is done for the case in which the threshold population is $T = \frac{n}{2}$. In section 4.2 an aggregated model is obtained while local stability and bifurcation analysis presented in section 4.3 and 4.4 respectively.

4.2 Aggregation

The model in Equation (3.12) is aggregated by considering that the market price evolves comparatively faster than the fish stock and the fishing effort. Fishing firms adjust to the market prices and the fishery conditions in order to avoid loses, recoup their investment and make profit. Approximate aggregation enables the reduction of the model in (3.12) to a system of two differential equations since the market price is assumed to be at a faster time scale than harvesting effort and population growth, see for instance Poggiale[29] and Segel[31]. The price in the harvesting effort is replaced with the nontrivial equilibrium values $p = p^*$. This is obtained when

$$\dot{p} = \alpha p(A - p - qnE) = 0, \tag{4.1}$$

so that p^* is given by,

$$p^* = A - qnE. \tag{4.2}$$

Thus, the second equation in (3.12) now becomes

$$\dot{E} = \beta E((A - qnE)qn - c). \tag{4.3}$$

We take $\beta = 1$, which is a maximum value in the range $0 \leq \beta \leq 1$ and may occur when the environmental conditions and harvesting are favorable for stock growth in the fishery. This aggregation reduces Equation (3.12) to ;

$$\begin{aligned} \dot{n} &= rn(\frac{n}{T} - 1)(1 - \frac{n}{k}) - qnE, \\ \dot{E} &= E((A - qnE)qn - c), \end{aligned} \tag{4.4}$$

a system of two differential equations we analyze in this study.

4.3 Local Stability Analysis

In this section, we seek for long term solutions of the model in (4.4) and carry out stability analysis to establish the nature of the equilibrium points.

4.3.1 Equilibrium Points

Solutions of the Model in (4.4) that do not change with time are found, these are the points where the system dynamics persist in time. In Model equations (4.4), the n nullclines are: $n = 0$ and $r(1 - \frac{n}{k})(\frac{n}{T} - 1) - qE = 0$ while the E nullclines are: $E = 0$ and $-c + qn(A - qnE) = 0$. The equilibrium points are basically the intersection of E and n nullclines, that is,

$$\begin{aligned} E_0 = (n_0, E_0) &= (0, 0) \\ E_1 = (n_1, E_1) &= (T, 0) \\ E_2 = (n_2, E_2) &= (k, 0) \\ E_3 = (n_3, E_3) &= (n^*, E^*) \end{aligned} \tag{4.5}$$

such that (n^*, E^*) is the solution of

$$\begin{aligned} E(n) &= \frac{r}{q}(1-\frac{n}{k})(\frac{n}{T}-1), \\ E(n) &= \frac{1}{qn}(A-\frac{c}{qn}). \end{aligned} \tag{4.6}$$

4.3.2 Stability Analysis

To analyse the stability of the equilibrium points of Equation (4.4), the system is linearized at the equilibrium points. The trace and determinant of the matrix of linearisation is studied, as model parameters are varied. The matrix of linearisation defines a linear map which is the best linear approximation of the function near the equilibrium point and it gives information about the local behaviour of the function.

Equation (4.4) can be expressed as

$$\begin{aligned} f(n,E): &= nr(1-\frac{n}{k})(\frac{n}{T}-1)-qnE, \\ g(n,E): &= -cE+qnE(A-qnE). \end{aligned} \tag{4.7}$$

The Jacobian matrix of Equation (4.4) is

$$J(n,E) = \begin{pmatrix} n(\frac{2r}{T}-\frac{3rn^2}{Tk}+\frac{2r}{k})-qE-r & -qn \\ qEA-2q^2nE^2 & -c+qnA-2q^2n^2E \end{pmatrix}.$$

At E_0 the Jacobian matrix is

$$J(0,0) = \begin{pmatrix} -r & 0 \\ 0 & -c \end{pmatrix},$$

whose eigenvalues are: $-r$ and $-c$. Since both are negative, the equilibrium point E_0 is a stable equilibrium point. This implies that any fish population and harvesting effort close to E_0 approaches E_0 in time. The population decays to zero due to sparsity which compromises reproduction, any small amount of harvesting makes the fish stock extinct. At E_1,

$$J(k,0) = \begin{pmatrix} r(1-\frac{k}{T}) & -qk \\ 0 & -c+qkA \end{pmatrix}.$$

Since $r \ll k$, $r \ll T$ and $k > T$, then $r(1-\frac{k}{T}) < 0$ if $k < \frac{c}{Aq}$, then $J(k,0)$ is a stable equilibrium point, for both eigenvalues are negative but if $k > \frac{c}{Aq}$, then $J(k,0)$ is a saddle point since one eigenvalue is negative and the other positive. Increased costs of the harvesting as compared to the maximum stock that can be supported in the fishery makes harvesting untenable hence stable state of equilibrium. In case the fish stock that can be supported by the fishery is larger enough than the costs, there is an increased harvesting effort which reduces the stock tremendously making the equilibrium point E_1 a saddle, which is unstable.

At the equilibrium E_2, the Jacobian matrix is given by

$$J(T,0) = \begin{pmatrix} r(1-\frac{T}{k}) & -qT \\ 0 & -c+qTA \end{pmatrix}.$$

Since $T < k$, $r(1-\frac{T}{k} > 0$ if $T < \frac{c}{Aq}$ then $J(T,0)$ is a saddle while if $T > \frac{c}{Aq}$, then $J(T,0)$ is unstable since both eigenvalues are positive. This equilibrium E_2 is unstable since any small perturbation from E_2 makes the fish stock to grow to the carrying capacity or decay to zero. It can be considered as a point where the fishery either declines to depensation and eventual species extinction or transits to a compensation population level where the population grows exponentially towards the carrying capacity .

At the Equilibrium E_3 denoted by (n^*, E^*) the Jacobian matrix is

$$J(n^*,E^*) := \begin{pmatrix} n^*(\frac{2r}{T} - \frac{3rn^{*2}}{Tk} + \frac{2r}{k}) - qE^* - r & -qn^* \\ qE^*A - 2q^2n^*E^{*2} & -c + qn^*A - 2q^2n^*E^* \end{pmatrix}. \quad (4.8)$$

To study the nature of the equilibrium point (n^*, E^*), we express c as a function of the equilibrium fish population n^* by equating the two equations in Equation (4.6) to have

$$\frac{r}{q}(1 - \frac{n}{k})(\frac{n}{T} - 1) = \frac{1}{qn}(A - \frac{c}{qn}). \tag{4.9}$$

On expansion and algebraic manipulation, Equation (4.9) yields

$$c(n^*) = \frac{rqn^{*4}}{Tk} - \frac{rqn^{*3}}{T} - \frac{rqn^{*3}}{k} + rqn^{*2} + Aqn^*. \tag{4.10}$$

With parameter values set at

$$r = q = A = 1,$$

Equation (4.10) takes its basic form whose geometric profiles portrays the dynamics of the fish stock and thus becomes,

$$c(n^*) = \frac{n^{*4}}{Tk} - \frac{n^{*3}}{T} - \frac{n^{*3}}{k} + n^{*2} + n^*. \tag{4.11}$$

We investigate this Equilibrium point (n^*, E^*) by having the threshold population T, as a factor of the fish population at a particular time n, at three different values. Exploitation of marine fish stock basing on **MSY** policy suggests that the stock is said to be sustainable if thirty to twenty percent of the stock that was initially present is in a fishery see Kar [19]. The threshold value T is a population value at which the fish stock population may decline into depensation and eventual species extinction due lack of social co-operative benefits associated to Allee effect or into compensation due to the harvesting of the stock. Analysis of each case is contained in the propositions and their proofs herein.

4.3.3 Case One: $T = \frac{n}{4}$

With $T = \frac{n}{4}$ and $q = 1$, Equation (4.10) becomes

$$c(n^*) = \frac{3rn^{*3}}{k} - 3rn^{*2} + An^*. \tag{4.12}$$

Figure 4.1 below shows how $c(n^*)$ depends on k.

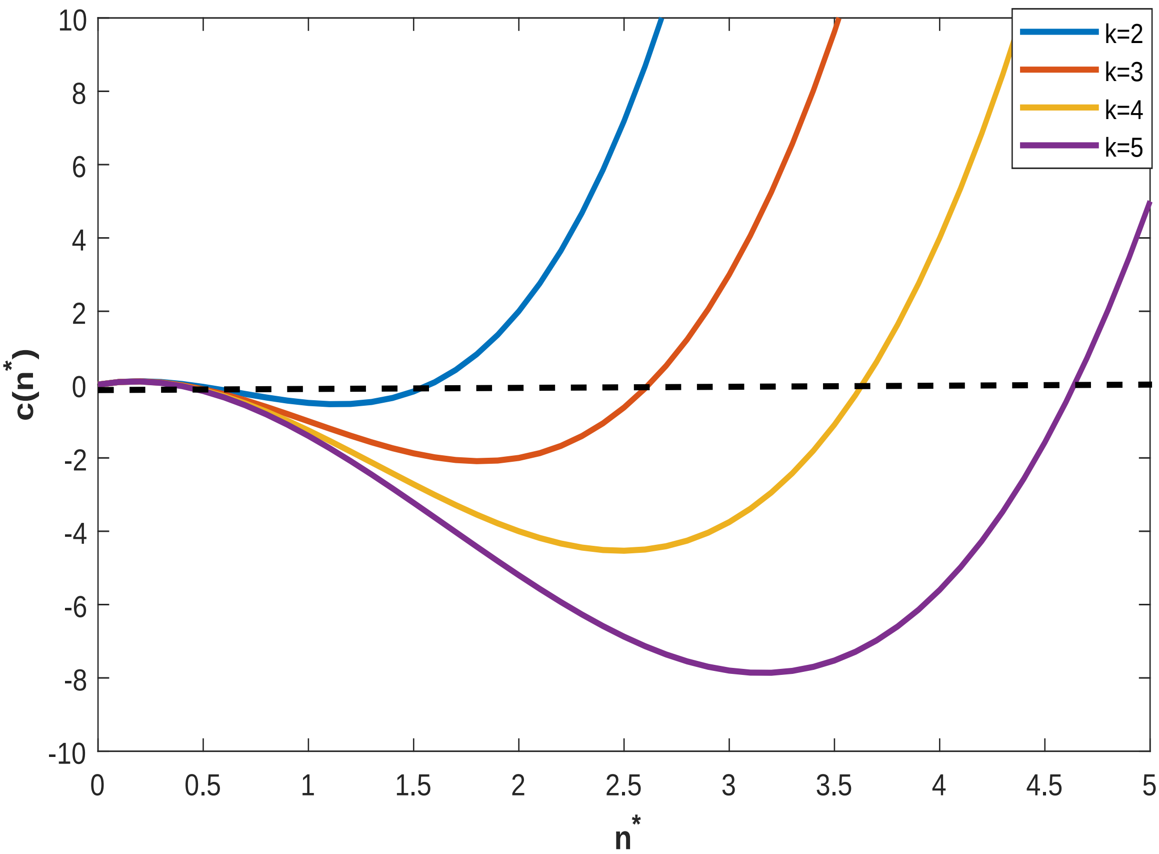

Figure 4.1: The function $c(n^*)$ plotted for $k = 2, 3, 4$ and 5. for $T = \frac{n}{4}$

Proposition 4.1

For $T = \frac{n}{4}$ there are three equilibrium points $(n_i{}^*, E_i{}^*)$ with $i = 1, 2, 3$ satisfying the Equilibrium condition

$$E^* = 3r(1 - \frac{n^*}{k})$$

such that $({n_1}^*, {E_1}^*)$ *and* $({n_2}^*, {E_2}^*)$ *are saddle points while* $({n_3}^*, {E_3}^*)$ *is a stable equilibrium point .*

Proof

The Jacobian matrix in Equation (4.8) for this value of T is

$$J(n^*, E^*) := \begin{pmatrix} \frac{-3rn}{k} & -qn \\ qEA - 2q^2nE^2 & -q^2n^{*2}E^* \end{pmatrix}.$$

The trace is $trJ(n^*, E^*) = -\frac{3rn}{k} - q^2n^{*2}E^* < 0$. The determinant $DetJ(n^*, E^*) = \frac{3r}{k}q^2n^{*3}E^* + q^2n^*E^*(A - 2qn^*E^*)$ which is equal to

$$DetJ(n^*, E^*) = q^2n^*E^*(\frac{3r}{k}n^{*2} - 2qn^*E^* + A.)$$

Using

$$E = \frac{3r}{q}(1 - \frac{n}{k}). \tag{4.13}$$

we have

$$qE^* = 3r(1 - \frac{n^*}{k})$$

which when substituted in the expression for determinant we obtain

$$DetJ(n^*, E^*) = q^2n^*E^*(\frac{9r}{k}n^{*2} - 6n^*r + A) = q^2n^*E^*\psi_1(n^*).$$

Where

$$\psi_1(n^*) = \frac{9r}{k}n^{*2} - 6n^*r + A.$$

The derivative of $c(n) = \frac{3rn^3}{k} - 3rn^2 + An$, with respect to n is $c\prime(n^*) = \frac{9r}{k}n^{*2} - 6rn^* + A$. The solution to this quadratic equation is

$$n^*_{1,2} = \frac{k}{3}\left(1 \pm \sqrt{1 - \frac{A}{kr}}\right). \tag{4.14}$$

$c\prime(n^*)$ is equal to ψ_1 and $sign\ DetJ(n^*, E^*) = sign\ \psi_1$. From the Figure 4.1 above, we have $sign\ c\prime(n) = sign\ DetJ(n^*, E^*) > 0$ when $n \subseteq [n_2, +\infty]$ and $sign\ c\prime(n) = sign\ DetJ(n^*, E^*) < 0$ when $n \subseteq [0, n_1] \bigcup [n_1, n_2]$. With $c(n_1) > 0$ and $c(n_2) < 0$ we have three equilibrium points $(n_1{}^*, E_1{}^*), (n_2{}^*, E_2{}^*), (n_3{}^*, E_3{}^*)$ such that $n_1{}^* < n_1 < n_2{}^* < n_2 < n_3{}^*$. For $(n_3{}^*, E_3{}^*)$, $n_2 < n_3{}^*$, $sign\ c\prime(n) = sign\ DetJ(n^*, E^*) > 0$ thus stable since $detJ(n_3{}^*, E_3{}^*) > 0$ and $trJ(n_3{}^*, E_3{}^*) < 0$. For $(n_1{}^*, E_1{}^*), (n_2{}^*, E_2{}^*)$, $n_1{}^* < n_1 < n_2{}^*$ $sign\ c\prime(n) = sign\ DetJ(n^*, E^*) < 0$ thus saddle equilibrium points since $trJ(n^*, E^*) < 0$ and $DetJ(n^*, E^*) < 0$.

4.3.4 Case Two: $T = \frac{n}{2}$

For $T = \frac{n}{2}$ and $q = 1$, Equation (4.10) becomes

$$c(n^*) = \frac{rn^{*3}}{k} - rn^{*2} + An^* \tag{4.15}$$

the graph of Equation (4.15) shows that as the parameter value k varies.

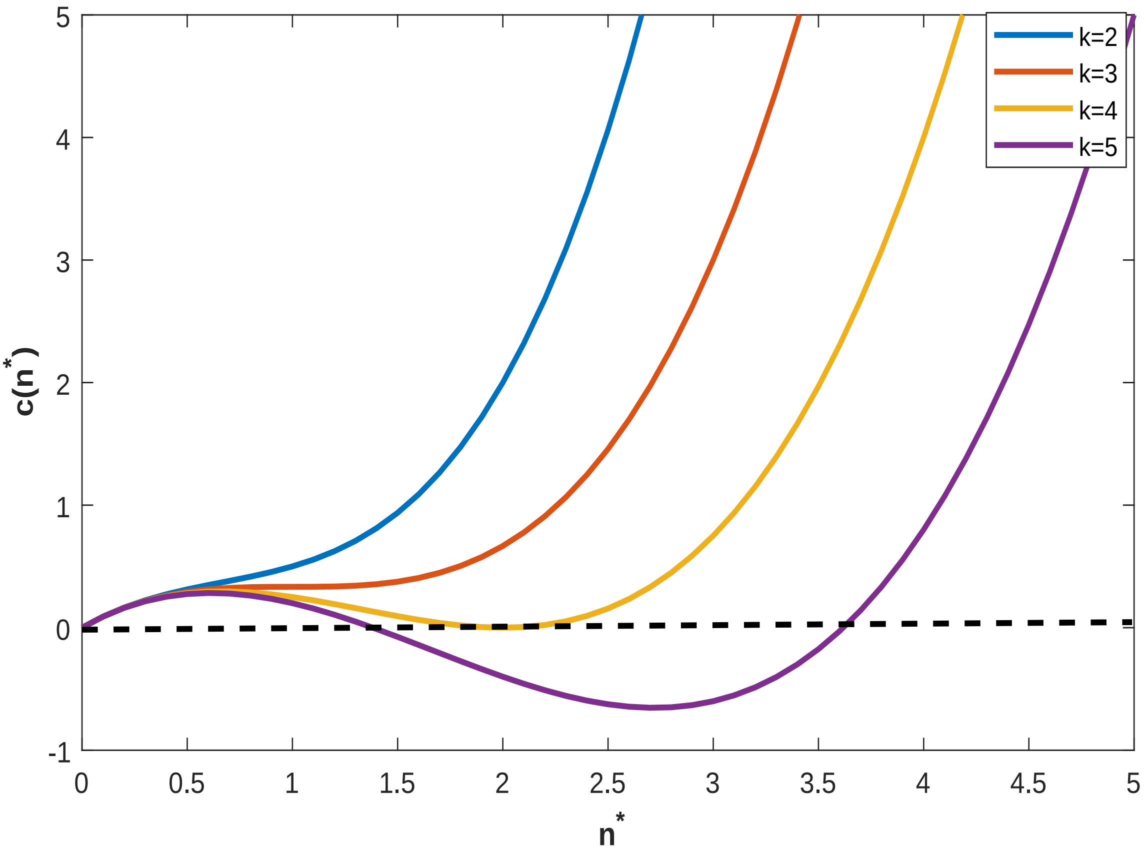

Figure 4.2: The function $c(n^*)$ plotted for $k = 2, 3, 4$ and 5. for $T = \frac{n}{2}$

the number of Equilibrium points is one for $k < 3$ and two more equilibrium points emerge for $k > 3$ as seen in Figure 4.2. In this case, $c\prime(n^*) = \frac{3r}{k}n^{*2} - 2rn^* + A$. The solution n^* for $c\prime(n^*) = 0$ are

$$n^*_{1,2} = \frac{k}{3}\left(1 \pm \sqrt{1 - \frac{3A}{kr}}\right). \tag{4.16}$$

Furthermore, The Jacobian matrix in Equation (4.8) with respect to these value of T is

$$J(n^*, E^*) = \begin{pmatrix} -\frac{r}{k}n^* & -qn^* \\ qE^*(A - 2qn^*E^*) & -q^2n^{*2}E^* \end{pmatrix}.$$

The trace and the determinant of $J(n^*, E^*)$ are:

$$trJ(n^*, E^*) = -\frac{r}{k}n^* - q^2n^{*2}E^* < 0$$

and

$$Determinant\ of\ J(n^*, E^*) = Det(J) := q^2n^*E^*(\frac{r}{k}n^{*2} + A - 2qn^*E^*)$$

respectively. Using

$$E^* = \frac{r}{q}(1 - \frac{n^*}{k}),$$

in $det(J)$, we obtain

$$Det(J) = q^2n^*E^*\psi_2(n^*),$$

where $\psi_2(n^*) := \frac{3r}{k}n^{*2} - 2rn^* + A$. Since $q^2n^*E^*$ is positive, the sign of $Det(J)$ depends on $\psi_2(n^*)$. For $Det(J)$, $\psi_2(n)$ and $(c\prime(n))$ have the same sign, we have:

$Det(J) > 0$ if $n\epsilon[0, n_1]\bigcup[n_2, +\infty]$,

$Det(J) < 0$ if $n\epsilon(n_1, n_2)$.

If $r < \frac{3A}{k}$, then $c\prime(n^*)$ is positive and $c(n^*)$ is monotonic increasing with complex roots. If $r > \frac{3A}{k}$ then there are two real zero's for $c\prime(n^*)$. If $r = \frac{3A}{k}$, the two real zero's coincide. At this point ${n^*}_{1,2} = \frac{k}{3}$, the parameters have further relationship as $c = \frac{k}{9}$ and $E^* = \frac{r}{q}(1 - \frac{n^*}{k}) = \frac{2A}{k}$. Further analysis distinguishes two different cases:

Proposition 4.2

For $0 < r < \frac{3A}{k}$ and $k > \frac{c}{Aq}$, $E_2 = (k, 0)$ is a saddle point and (n^*, E^*) is a positive stable equilibrium point.

Proof

If $0 < r < \frac{3A}{k}$ in Equation (4.15), the sign of $c\prime(n^*)$ is positive. Moreover, $c\prime\prime(n^*) = \frac{6rq}{k}n^* - 2rq$ thus $n^* = \frac{k}{3}$ is a point of inflection. We have $c(k) = qAk$ but since c is strictly increasing and may take positive or negative values depending on k, we consider $c(k) = qAk - c$ and as $lim_{n\to+\infty}c(n) = +\infty$, we conclude that c vanishes at a unique point n^*, thus a unique equilibrium point (n^*, E^*) is obtained. If $k < \frac{c}{Aq}$, then $c(k) < 0$ and c vanishes at a value $n^* > k$,which corresponds to a negative effort equilibrium ($E^* < 0$). In this case, the equilibrium point E_2 is a stable equilibrium but (n^*, E^*) does not present any interest since it is corresponding to unrealistic negative fishing effort, but if $k > \frac{c}{Aq}$ then $c(k) > 0$ and c vanishes at a value $n^* < k$, with a positive effort equilibrium $E^* > 0$. In this case E_2 is a saddle point and (n^*, E^*) is the unique positive stable since $detJ({n_3}^*, {E_3}^*) > 0$ and $trJ({n_3}^*, {E_3}^*) < 0$.

Proposition 4.3

For $0 < \frac{3A}{k} < r$, $E_i := (n_i^*, E_i^*)$ for $i = 1, 2, 3$ three positive equilibrium points exist with three subcases:

1. *If $c(n^*) < 0$, $c(n_1^*) < 0$, there is a unique positive and stable equilibrium point(n^*, E^*);*

2. *If* $c(n_1^*) > 0$ *and* $c(n_2^*) > 0$ *, a unique positive and stable equilibrium point is obtained* (n^*, E^*)*;*

3. *If* $c(n_1^*) > 0$ *and* $c(n_2^*) < 0$ *, three positive equilibrium points* (n_i^*, E_i^*) *for* $i = 1, 2, 3$ *exist with* (n_1^*, E_1^*) *and* (n_3^*, E_3^*) *stable while* (n_2^*, E_2^*) *is a saddle equilibrium point.*

Proof

If $0 < \frac{3A}{k} < r$ in Equation (4.15), $c\prime$ vanishes at two values n_1 and n_2 given by

$$0 \leq n_1 = \frac{k}{3}\left(1 - \sqrt{1 - \frac{3A}{rk}}\right) < \frac{k}{3},$$

and

$$\frac{k}{3} < n_2 = \frac{k}{3}\left(1 + \sqrt{1 - \frac{3A}{rk}}\right) < k.$$

Recall that $\lim_{n^* \to +\infty} c(n^*) = +\infty$. As $c(n_1^*)$ and $c(n_2^*)$ can have positive or negative signs,so for subcase 1, with $c(n^*) < 0$, $c(n_1^*) < 0$ and $n^* > n_1$, $det(J) > 0$ and $tr(J) < 0$ thus a stable equilibrium point (n^*, E^*). For Subcase 2, since $c(n_1^*) > 0$ and $c(n_2^*) > 0$, with $n^* < n_1^*$, $det(J) > 0$ and $tr(J) < 0$ thus a stable equilibrium point (n^*, E^*). Finally for Subcase 3, given that $n_1^* < n_1 < n_2^* < n_2 < n_3^*$ is satisfied, (n_1^*, E_1^*) and (n_3^*, E_3^*) are stable since $det(J) > 0$ and $tr(J) < 0$ while (n_2^*, E_2^*) is a saddle equilibrium point since $det(J) < 0$ and $tr(J) < 0$.

4.3.5 Case Three : $T = \frac{3n}{4}$

With $T = \frac{3n}{4}$ and $q = 1$, Equation (4.10) becomes

$$c(n) = \frac{rn^3}{3k} - \frac{r}{3}n^2 + An. \tag{4.17}$$

Figure 4.3 shows how $c(n)$ depends on k.

The derivative of Equation ((4.17)) with respect to n is $c\prime(n^*) = \frac{r}{k}n^{*2} - \frac{2}{3}rn^* + A$.

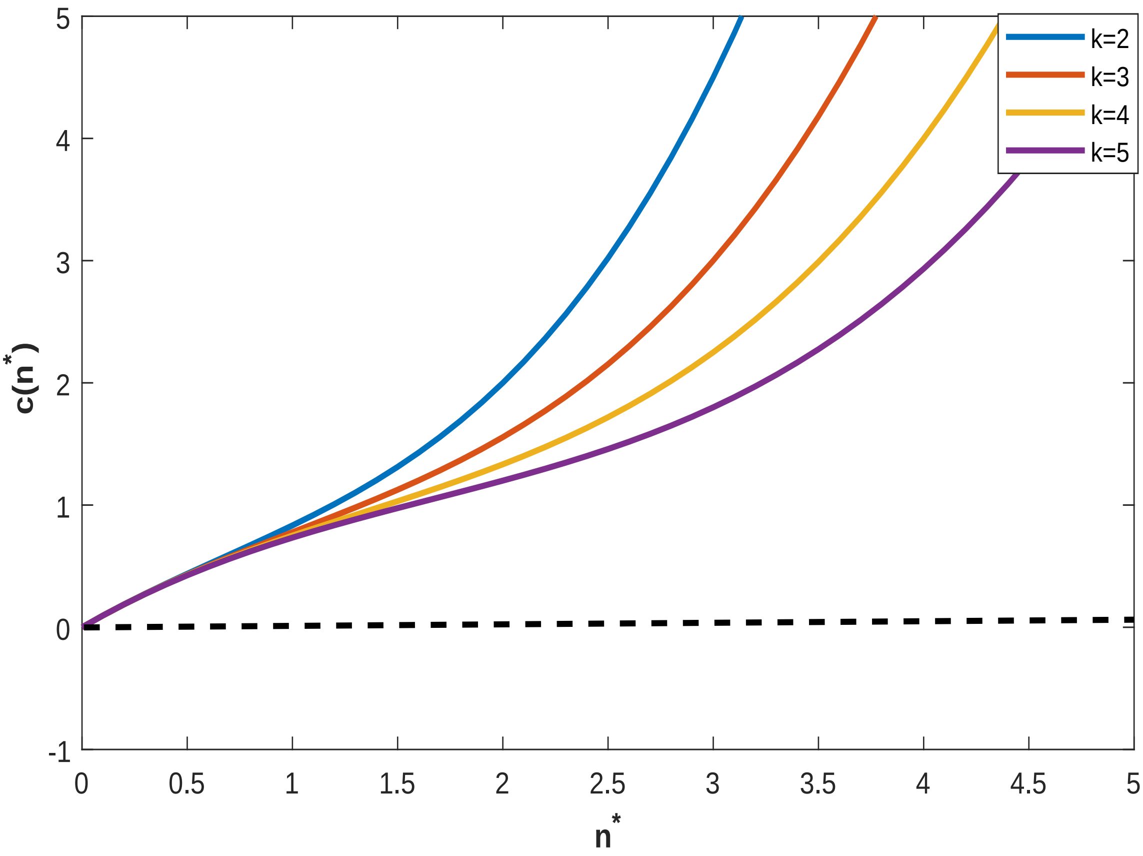

Figure 4.3: The function $c(n^*)$ plotted for $k = 2, 3, 4$ and 5. for $T = \frac{3n}{4}$

Which vanishes at

$$n^*_{1,2} = \frac{2k}{3}\left(1 \pm k\sqrt{\frac{1}{9} - \frac{A}{kr}}\right). \tag{4.18}$$

The Equation (4.18) has complex roots not relevant to a realistic fish population.

Proposition 4.4

For $T = \frac{3n}{4}$ there is one equilibrium point (n^*, E^*) satisfying the equilibrium condition

$$E^*(n) = \frac{r}{3}(1 - \frac{n^*}{k}) \tag{4.19}$$

which is stable.

Proof

The Jacobian matrix in Equation (4.8) corresponding to this value of T is

$$J(n^*, E^*) := \begin{pmatrix} \frac{-rn^*}{3k} & -qn^* \\ qE^*A - 2q^2n^*E^{*2} & -q^2n^{*2}E^* \end{pmatrix}.$$

The trace is $trJ(n^*, E^*) = -\frac{rn^*}{3k} - q^2n^{*2}E^* < 0$. The determinant $DetJ(n^*, E^*) = \frac{r}{3k}q^2n^{*3}E^* + q^2n^*E^*(A - 2qn^*E^*)$, which is equal to

$$DetJ(n^*, E^*) = q^2n^*E^*(\frac{r}{3k}n^{*2} - 2qn^*E^* + A).$$

Using

$$E = \frac{r}{3q}(1 - \frac{n^*}{k}), \tag{4.20}$$

in the expression of the determinant, we obtain

$$Det\ J(n^*, E^*) = q^2n^*E^*(\frac{r}{k}n^{*2} - \frac{2}{3}n^*r + A) = q^2n^*E^*\psi_3(n^*).$$

Where

$$\psi_3(n^*) = \frac{r}{k}n^{*2} - \frac{2}{3}n^*r + A.$$

Figure 4.3 shows that $c\prime(n) > 0$ and $c(n)$ is monotone increasing thus $c(n)$ vanishes at a unique value $n^* < k$. Since the sign of the determinant is the same as the sign of ψ_3 which is also the sign of $c\prime$, the equilibrium point (n^*, E^*) is stable since $sign\ c\prime(n) = sign\ DetJ(n^*, E^*) > 0$. The stable equilibrium points in the above propositions corresponds to solution values for the fish population and harvesting in which any initial population and harvesting effort close to them will eventual move close to the points in time whereas the saddle equilibrium points are the solution values for the system in which any initial fish population and harvesting effort close to the point will move away from the point in time. This is mainly attributed to the cost of harvesting which dictates whether harvesting is economically tenable or not. It is the magnitude of harvesting effort which determines the fish stock levels in the fishery.

4.4 Bifurcation analysis

Bifurcation is the study of the changes in the qualitative structure of a given family, often, a dynamical system. Bifurcation is said to occur when a smooth change in the parameter of a system causes a sudden qualitative change in its behaviour, see Blanchard E. P *et al.*[9].

In this section, we examine the model for the case in which $T = \frac{n}{2}$, and k as the bifurcation parameter. For $k = 2, 3$, there is one equilibrium point and when $k = 4, 5$, two more equilibrium points emerge depicting a bifurcation. The model in (4.4) when T is replaced with $T = \frac{n}{2}$, it becomes

$$\begin{aligned} \dot{n} &= n(r(1 - \frac{n}{k}) - qE), \\ \dot{E} &= E(-c + qn(A - qnE)). \end{aligned} \tag{4.21}$$

4.4.1 Non dimensionalisation

We make Equation (4.21) dimensionless by making the following transformations:

$$\upsilon = \sqrt{q}n, \varepsilon = \frac{\sqrt{q}}{A}E, \tau = \sqrt{q}At, \tag{4.22}$$

and introducing the parameters;

$$\rho = \frac{r}{A\sqrt{q}}, \gamma = \frac{c}{A\sqrt{q}}, \kappa := \sqrt{q}k. \tag{4.23}$$

By chain rule, the derivative of n is given thus:

$$\dot{n} = \frac{dn}{d\upsilon}.\frac{d\upsilon}{d\tau}.\frac{d\tau}{dt} = A\frac{d\upsilon}{d\tau}, \tag{4.24}$$

using (4.22), (4.23) and (4.24) in the first equation of (4.21) we get

$$A\frac{d\upsilon}{d\tau} = \frac{\upsilon}{\sqrt{q}}(\rho A\sqrt{q}(1 - \frac{\upsilon}{\kappa}) - \frac{\upsilon^2}{n^2}.\frac{\varepsilon A}{\sqrt{q}}), \tag{4.25}$$

which upon simplification yields:

$$\dot{\upsilon} = \upsilon(\rho(1 - \frac{\upsilon}{\kappa}) - \varepsilon), \tag{4.26}$$

where the dot "." indicates differentiation with respect to τ. Similarly, the derivative of E is given in dimensionless terms by :

$$\dot{E} = \frac{dE}{d\varepsilon}.\frac{d\varepsilon}{d\tau}.\frac{d\tau}{dt} = A^2\frac{d\varepsilon}{d\tau},$$

which upon substitution in the second equation of (4.21) and the use of the dimensionless variables and parameters in Equations (4.22) and (4.23), we obtain:

$$A^2\frac{d\varepsilon}{d\tau} = \frac{\varepsilon A}{\sqrt{q}}(-\gamma A\sqrt{q} + q\frac{\upsilon}{\sqrt{q}}(A - q\frac{\upsilon}{\sqrt{q}}\frac{\varepsilon A}{\sqrt{q}})), \tag{4.27}$$

that simplifies to:

$$\dot{\varepsilon} = \varepsilon(-\gamma + \upsilon(1 - \varepsilon\upsilon)) \tag{4.28}$$

The model in Equation (4.21) expressed in dimensionless terms becomes:

$$\begin{aligned} \dot{\upsilon} &= \upsilon(\rho(1 - \frac{\upsilon}{\kappa}) - \varepsilon), \\ \dot{\varepsilon} &= \varepsilon(-\gamma + \upsilon(1 - \varepsilon\upsilon)), \end{aligned} \tag{4.29}$$

Three parameters; κ, ρ, γ remain. These parameters are interpreted as follows; in case $\rho \ll 1$, and $\gamma \ll 1$, then $A\sqrt{q} \gg r$ and $A\sqrt{q} \gg c$ where we have demand driven over-exploitation of the fish resource. If $\gamma \gg 1$ and $\rho \gg 1$ then it follows that $A\sqrt{q} \ll r$ and $A\sqrt{q} \ll c$ which will lead to under-exploitation of the fish resource.

Comparison of dimensionless model in Equation (4.29) to Equation (4.21), shows that if we set $r = q = A = 1$ in (4.21), we obtain Model equation in (4.29) with $\upsilon = n$, $\varepsilon = E$, $\rho = r$, $\gamma = c$ and $\kappa = k$, thus, we use initial parameters k, c and r.

Bifurcation will show us the long-term dynamic behaviour of the aggregated model. We shall show that there is a value of the bifurcation parameter $k =: k_0$ where the system in (4.29) undergoes a saddle-node bifurcation showing the co-existence of two stable equilibria separated by a saddle, whereby the fish population and the fishing effort varies with k. This is done by stating and proofing Proposition 4.5 and describing two bifurcation diagrams that show the number and type of stability of points of equilibria as k is varied.

Proposition 4.5 *For $n > 2c$, there is a value of $k =: k_0$ where the system in Equation (4.21) undergoes saddle - node bifurcation as the fish population and the fishing effort dynamics vary with the carrying capacity. Furthermore, for $k < k_0$, there are only two equilibria while when $k > k_0$, two more equilibria emerge, one stable and the other unstable.*

Proof

Using Equations (4.21) and

$$\begin{aligned} E(n) &= \frac{r}{q}(1 - \frac{n}{k}) \\ E(n) &= \frac{1}{qn}(A - \frac{c}{qn}), \end{aligned} \tag{4.30}$$

the second equation in Equation (4.30) gives

$$qE = \frac{1}{n}(A - \frac{c}{qn}). \tag{4.31}$$

Equation (4.31) when substituted in the first equation of Equation (4.21) it yields

$$\dot{n} = n(r - \frac{rn}{k} - \frac{A}{n} + \frac{c}{qn^2}), \tag{4.32}$$

with $r = q = A = 1$ as earlier seen, we obtain

$$\dot{n} = n\phi(n,k) =: \Phi(n,k),$$

where

$$\phi(n,k) := 1 - \frac{n}{k} - \frac{1}{n} + \frac{c}{n^2}. \tag{4.33}$$

Clearly, $n = 0$ and the curve $\phi(n,k) = 0$ gives the equilibrium points. For the stability of the equilibria points $\phi(n,k) = 0$, we have

$$\Phi'(n,k) = \phi(n,k) + n\phi'(n,k),$$

when $\phi(n,k) = 0$, we obtain

$$\Phi'(n,k) = n\phi'(n,k),$$

where the prime indicates differentiation with respect to n. There is stability when $\phi'(n,k) < 0$ and instability when $\phi'(n,k) > 0$. Since $\phi'(n,k)$ is continuous for $n > 0$, we have a change in stability at $\phi'(n,k) = 0$; that is,

$$\phi'(n,k) = \frac{-1}{k} - \frac{2c}{n^3} + \frac{1}{n^2} = 0,$$

or

$$-n^3 - 2ck + nk = 0,$$

and find that

$$k = k_0 := \frac{n^3}{n - 2c}, \tag{4.34}$$

as the value of k where a bifurcation occurs.

To be able to indicate the nature of stability in the bifurcation diagram in Figure 4.4 obtained using (4.33), consider

$$\dot{n} = n\phi(n,k) = 0.$$

The curve $\phi(n,k) = 0$ defines equilibrium point (n^*, k^*). Clearly

$$\frac{d\phi}{dk} = \frac{n*}{k^2} \mid_{(n^*,k^*)} > 0,$$

thus by the Implicit Function Theorem, there exists

$$\phi(n, k(n)) = 0, \tag{4.35}$$

$k(n)$ defined in the neighbourhood of (n^*, k^*) with $k(n^*) = k^*$ as smooth as $\phi(n, k)$. Differentiating Equation (4.35) with respect to n, we get

$$\frac{d\phi}{dn} + \frac{d\phi}{dk}\frac{dk}{dn} = 0$$

and hence

$$\frac{d\phi}{dn} = -\frac{d\phi}{dk}\frac{dk}{dn},$$

from which we can see that

$$sign(\frac{d\phi}{dn}) = -sign(\frac{dk}{dn}).$$

Hence the nature of stability in the bifurcation diagram in Figure 4.4, where the variation of k, beyond $k =: k_0$ as earlier defined in Equation (4.34). leads to creation of two more additional equilibrium solutions. This is a bifurcation with the fish population as the variable and the carrying capacity k as the bifurcation parameter. As earlier seen in simulation in Figure 3.4, a harvesting reduces the carrying capacity of the fishery, thus as the k varies, there is a value of k as given in Equation (4.34),where a decrease in the fish population in an under-exploited fishery causes the stable population value $n_3{}^*$ to attain the saddle population value $n_2{}^*$ earlier presented in proposition 4.3 (iii).

For the variation of the fishing effort with the carrying capacity as the bifurcation

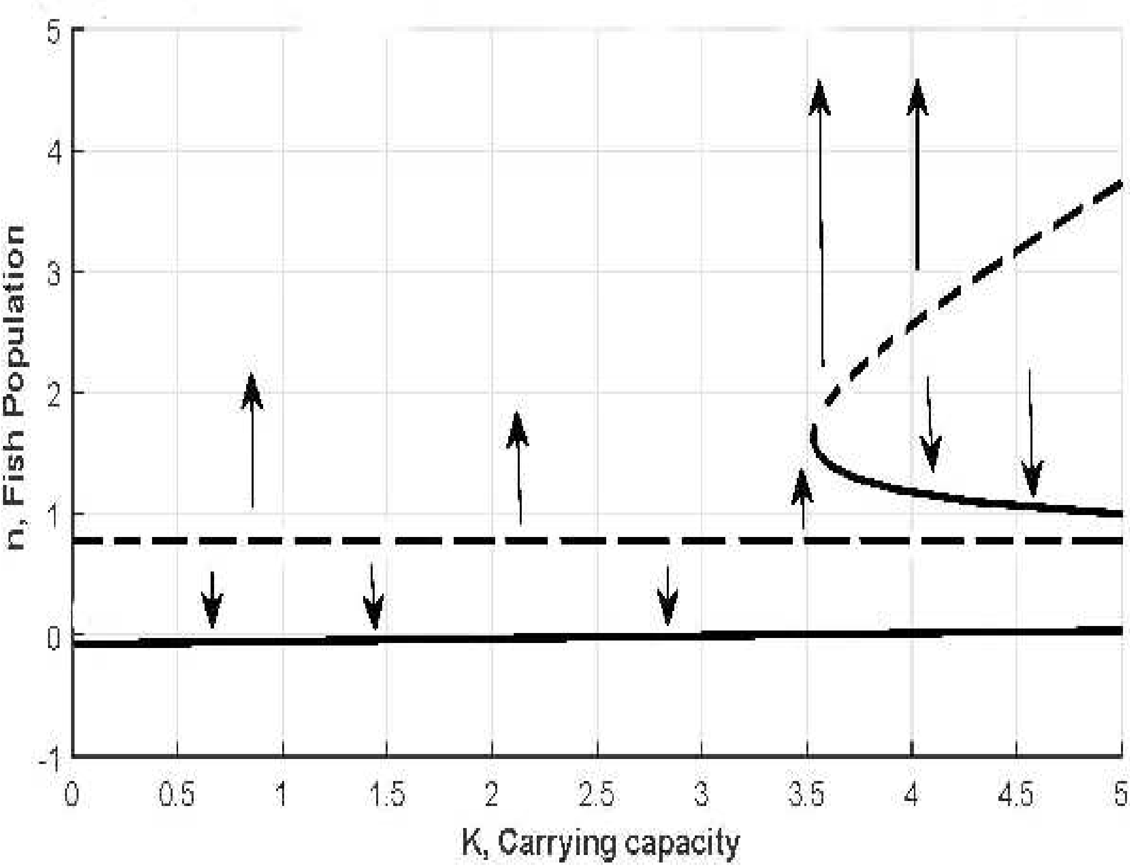

Figure 4.4: One Parameter bifurcation diagram for fish population with k as parameter

parameter, we obtain

$$\dot{E} = E(-c + n - n^2 E), \tag{4.36}$$

where $q = A = 1$ in the second equation of Equation (4.21). Similarly, we also obtain

$$n = k(1 - E), \tag{4.37}$$

from the first equation in Equation (4.30) such that

$$k = k_0 = \frac{k^3(1-E)^3}{k(1-E) - 2c}, \tag{4.38}$$

is the bifurcation value. Further aggregation in which Equation (4.37) is substituted in Equation (4.36) yields

$$\dot{E} = E\Theta(E,k) = \Psi(E,k),$$

where

$$\Theta(E,k) = -c + k - kE - k^2E + 2k^2E^2 - k^2E^3. \tag{4.39}$$

Clearly, $E = 0$ and the curve $\Theta(E,k) = 0$ gives equilibrium points. For the stability of the equilibrium points $\Theta(E,k) = 0$, we have

$$\Psi'(E,k) = \Theta(E,k) + E\Theta'(E,k),$$

and hence

$$\Psi'(E,k) = E\Theta'(E,k),$$

if evaluated at $\Theta(E,k) = 0$. The prime indicates differentiation with respect to E. There is stability when $\Theta'(E,k) < 0$ and instability when $\Theta'(E,k) > 0$. Since $\Theta'(E,k)$ is continuous with $E > 0$, there is a change in stability at $\Theta'(E,k) = 0$. To indicate the nature of stability, consider

$$\dot{E} = E\Theta(E,k) = 0.$$

The curve $\Theta(E,k) = 0$ defines equilibrium points (E^*, k^*).

$$\frac{d\Theta}{dk} = 1 - E - 2kE + 4kE^2 - 2kE^3,$$

where it is seen that $\frac{d\Theta}{dk} \mid_{(E^*,k^*)} < 0$ for $E^* > 0$. Thus, by the Implicit Function Theorem, there exists

$$\Theta(E, k(E)) = 0, \tag{4.40}$$

$k(E)$ defined in the neighbourhood of (E^*, k^*) with $k(E^*) = k^*$ as smooth as

$\Theta(E, k)$. Differentiating Equation (4.40) with respect to E, we obtain

$$\frac{d\Theta}{dE} + \frac{d\Theta}{dk}\frac{dk}{dE} = 0$$

thus

$$\frac{d\Theta}{dE} = -\frac{d\Theta}{dk}\frac{dk}{dE}.$$

Since $\frac{d\Theta}{dk} < 0$, it is evident that

$$sign(\frac{d\Theta}{dE}) = sign(\frac{dk}{dE}),$$

as seen in Figure 4.5 obtained using Equation (4.39). The stability changes at $\Theta'(E, k) = 0$, hence the nature of stability shown. There is existence of only two equilibrium solutions before the bifurcation parameter k passes through the critical value $k = k_0$ defined in (4.38) where two more equilibrium solutions are generated, one being stable and the other unstable as seen in Figure 4.5. Harvesting, reduces the fish population in the fishery which varies with the carrying capacity. As harvesting varies, there is a value $k = k_0$ defined in (4.38) where the equilibrium Value E^* decreases to make the equilibrium fish population n^* increase hence a change from unstable equilibrium point to a stable equilibrium point as presented in proposition 4.3(iii).

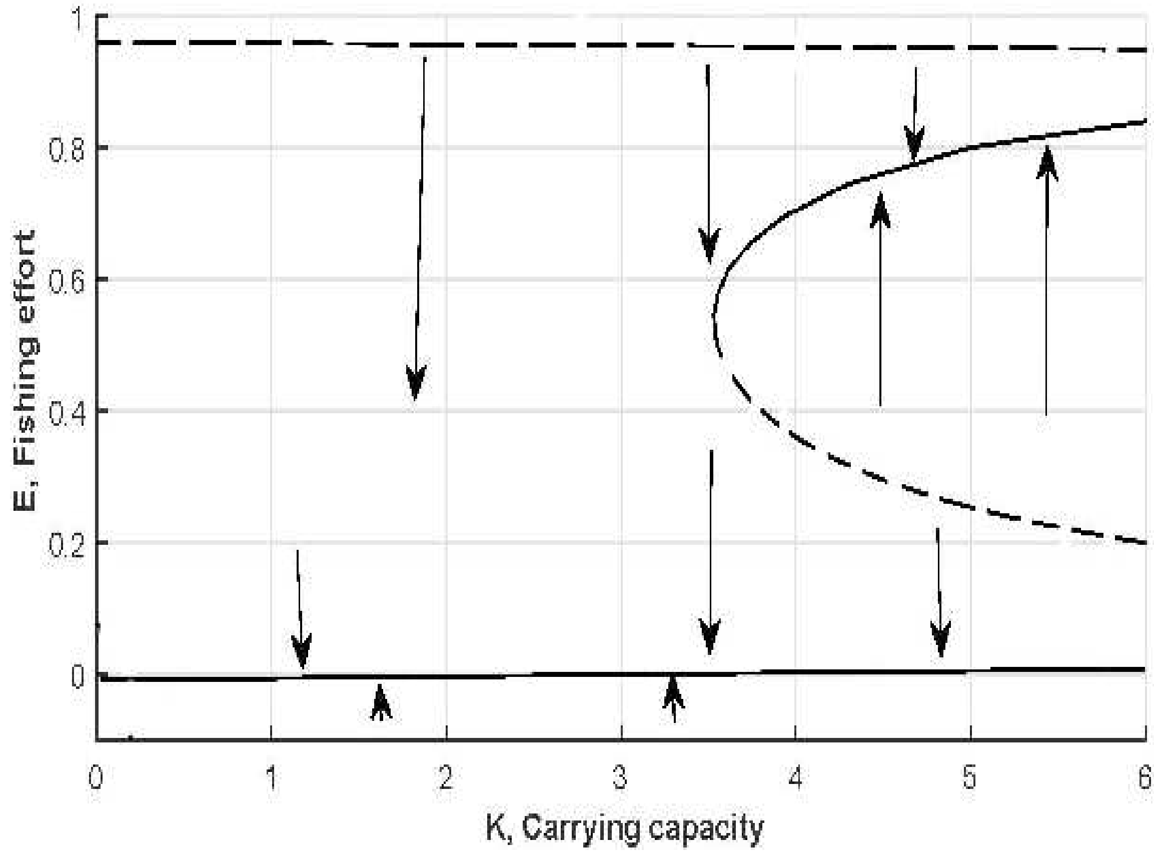

Figure 4.5: One Parameter bifurcation diagram for Fishing Effort with k as parameter

CHAPTER FIVE

DISCUSSION, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

This chapter presents a discussion of results obtained in chapter 4, in relation to the commercial marine fishery at the Kenyan coastline, where tuna species is dominant. Section 5.2 is the discussion while conclusion and recommendation is contained in section 5.3 and 5.4 respectively.

5.2 Discussion

In this study, we have presented a mathematical model of a fishery with Allee effect in the population growth equation. Thus we have considered the dynamics of the fishery at low population levels which addresses fears of species extinction. Moreover, with consideration of the harvesting effort which is determined by the market price of landed fish, economic aspects have also been incorporated. The market price is considered to evolve faster than the harvesting effort and the population growth since prices are determined on a day to day basis as investors strive to recoup their investment. Rapid evolution in the market price is used to aggregate the model from a system of three equations to a system of two equations by using its equilibrium value in the harvesting effort equation. The results obtained with consideration of Allee effects in the population growth equation differ significantly from results of models without Allee effects in Auger *et al.*[4], Mchich *et al.* [28] and Makwata H. *et al.* [22]. Analysis realised four equilibrium points, namely; $(0,0),(T,0),(k,0)$ and $(n_i{}^*, E_i{}^*)$ for $i = 1,2,3$. The dilemma of co-existence of two fishery states which do not support sustainable fish resource exploitation in Auger *et al.*[4], and Makwata H. *et al.* [22] is addressed by the threshold population value

$T = \frac{n}{4}$ which gives co-existence of three equilibrium points, two saddle equilibrium points and a stable equilibrium point.

Simulations of the population growth equation

$$\dot{n} = rn(\frac{n}{T} - 1)(1 - \frac{n}{K}) - qnE, \tag{5.1}$$

is performed with the population starting at varying initial values as shown in Figure 3.3 and 3.4. Two situations are displayed; a case where there is zero harvesting and the case where a minimal harvesting is allowed. These simulation curves depict a fishery where harvesting is not a major priority but the fishery can support recreational economic activity of fishing sports, with species like Yellow fin tuna, Skipjack tuna and Black marlins often catched and released in the sport. Thus, investments can be made in tourism industry to develop fishing sports along the Kenyan marine coastline to enhance returns from blue economy.

Analysis of the equilibrium (n^*, E^*) at three different values of $T = \frac{n}{4}, T = \frac{n}{2}$ and $T = \frac{3n}{4}$ shows one equilibrium or co-existence of three positive equilibria. The stable equilibrium corresponds to the fish stock being maintained at high levels but with low economic activity This is obtained when $T = \frac{3n}{4}$. This is the current situation of the Kenya commercial marine fisheries where fishing activity is mainly traditional. The case of two stable equilibria co-existing with a saddle between is reasonably observed in most commercial fisheries, it is obtained when $T = \frac{n}{2}$. This occurs when the condition $rk < 3A$ holds. The condition implies that the fish species under exploitation has rapid reproduction and a large carrying capacity, a very common case with most commercial species. The co-existence of these three equilibria is due to an increase in harvesting costs due to increased fuel prices, taxes and wages. The two stable equilibria correspond to an over-exploited and an under-exploited fish stock. Immediate history of the fishery may play a core role in its evolution and in its current state. The transient dynamics determines the closeness of the fishery to either of the two stable equilibria depending on the initial conditions. Thus the fishery can be found in two distinct situations: Over-exploited or under-exploited. However, it is not possible to decide wether the fishery should persist in either of the situations. Over-exploitation has a risk of species extinction whereas under-exploitation has low economic returns not economically viable. Kenya commercial marine fishery is in an under-exploited state but with increased economic exploitation, it can transit to the over-exploited state. The case of co-existence of three positive equilibria co-existing, two being saddle and one being a stable obtained when $T = \frac{n}{4}$ corresponds to a fishery in which there are management practices in which the fish resource is exploited in a sustainable manner. When the management practices are enforced, the fish stock is maintained at sustainable equilibrium value ${n_2}^*$, with a possibility of sliding to vulnerable levels ${n_1}^*$ if the management practices are not enforced or huge levels of little economic activity ${n_3}^*$. In case the fishery is found in over-exploited state, the only management practice is to enforce a total ban on exploitation for some time for the fish stock to recover, a management practice commonly practised with threatened fresh water species.

5.3 Conclusion

Kenya commercial marine fishery is currently under - exploited, for economic benefits, an aggressive harvesting strategy should be in place with appropriate investments. However, best fishery management practices need to be formulated depending on the species census data which will enhance setting of sustainable fishing quotas and enforcement of these regulation by relevant authorities. The over-exploitation equilibrium in this study do not cause an immediate extinction but to a population density $n_1{}^*$, of the fish resource. Kenya can maintain a huge economic activity but this can be done in a durable way in which estimates of the amplitude of environmental variations which may cause extinction is done and maintaining the equilibrium fish density n_1 at a sufficient density so that the fish resource can recover incase of a major modification of environmental conditions.

5.4 Recommendations

From the analysis, sustainable harvesting of Kenyan coastal commercial fish stocks can be achieved with enhanced monitoring and surveillance. Kenya has made advanced efforts in surveillance with the establishment of the Kenya Coast Guard Service. However, more efforts need to be made in monitoring where records of all catches are made and research on fish stocks made so that more analysis to fix parameter like capturability which sets appropriate fishing quotas at varying times.

Most commercial marine fish species are highly migratory determined by availability of prey and movement to reproduction grounds. Similarly, most fishery harvesting vessels will move to fishery patches with high stock density. Kenya marine fish species and its harvesting strategy can not be an exception. A fishery model considering density dependent migratory rates both in the fish stock and in the harvesting vessels, as they move between the fishing zones is a feasible future research problem of interest.

Appendix A

APPENDIX: MATLAB R2017b COMMANDS FOR FIGURES

1.1 Figure 3.1

$r = 0.05;$

$Tot - time = 100;$

$dt = 0.01;$

$t = (0 : dt : Tot - time;$

$n = Zeros(size(t));$

$tot - steps = Tot - time/dt + 1$

$ic(1) = 0;$

$ic(2) = 1;$

$ic(3) = 2;$

$forj = 1 : 3$

$forj = ic(j);$

$n(1) = ic(j);$

$fori = 1 : tot_steps - 1$

$n(i + 1) = n(i) + dt * (r * n(i));$

end

figure(1)

$if(j == 1)$

plot(t,n,‘r’)

elseif $(j == 2)$

plot(t,n,‘b’)

else

plot(t,n,'g')

end

end

1.2 Figure 3.2

$r = 0.05;$

$Tot - time = 100;$

$dt = 0.01;$

$t = (0 : dt : Tot - time;$

$n = Zeros(size(t));$

$k = 500;$

$tot - steps = Tot - time/dt + 1$

$ic(1) = 0;$

$ic(2) = 1;$

$ic(3) = 2;$

$forj = 1 : 3$

$forj = ic(j);$

$n(1) = ic(j);$

$fori = 1 : tot_steps - 1$

$n(i + 1) = n(i) + dt * (r * n(i) * (1 - n(i)/k);$

end

figure(1)

$if(j == 1)$

plot(t,n,'r')

$elseif(j == 2)$

plot(t,n,'b')

else

plot(t,n,‘g’)

end

end

1.3 Figure 3.3

$r = 0.05;$

$T = 100;$

$k = 500;$

$E = 0;$

$Tot - time = 100;$

$dt = 0.01;$

$t = (0 : dt : Tot - time);$

$n = Zeros(size(t));$

$tot - steps = Tot - time/dt + 1;$

$ic(i) = 520;$

$ic(2) = 450;$

$ic(3) = 300;$

$ic(4) = 120;$

$ic()5) = 50;$

$forj = 1 : 5$

$n(1) = ic(j);$

$fori = 1 : tot - steps - 1$

$n(i + 1) = n(i) + dt * (-r * n(i) * (1 - n(i)/T) * (1 - n(i)/k) - E * n(i));$

end

figure (1)

hold on

$if(j == 1)$

plot(t,n,'r')

$elseif(j == 2)$

plot(t,n,'b')

$elseif(j == 3)$

plot(t,n,'g')

$elseif(j == 4)$

plot(t,n,'y')

else plot(t,n,'k')

end

end

end

end

end

$legend(`520', `450', `300', `120', `50')$

1.4 Figure 3.4

$r = 0.05;$

$T = 100;$

$k = 500;$

$E = 0.5;$

$Tot - time = 100;$

$dt = 0.01;$

$t = (0 : dt : Tot - time);$

$n = Zeros(size(t));$

$tot - steps = Tot - time/dt + 1;$

$ic(i) = 520;$

$ic(2) = 450;$

$ic(3) = 300;$

$ic(4) = 120;$

$ic()5) = 50;$

$forj = 1 : 5$

$n(1) = ic(j);$

$fori = 1 : tot - steps - 1$

$n(i + 1) = n(i) + dt * (-r * n(i) * (1 - n(i)/T) * (1 - n(i)/k) - E * n(i));$

end

figure (1)

hold on

$if(j == 1)$

plot(t,n,'r')

$elseif(j == 2)$

plot(t,n,'b')

$elseif(j == 3)$

plot(t,n,'g')

$elseif(j == 4)$

plot(t,n,'y')

else plot(t,n,'k')

end

end

end

end

end

$legend(‘520', ‘450', ‘300', ‘120', ‘50')$

1.5 Figure 4.1

$n = 0 : 0.1 : 5;$

$c = 3 * \frac{(n.^3)}{2} - n.^2 + n;$

$c_1 = 3 * \frac{(n.^3)}{3} - n.^2 + n;$

$c_2 = 3 * \frac{(n.^3)}{4} - n.^2 + n;$

$c_3 = 3 * \frac{(n.^3)}{5} - n.^2 + n;$

$\text{plot}(n, c, n, c_1, n, c_2, n, c_3)$

$xlabel('n*')$

$ylabel('c(n*)')$

1.6 Figure 4.2

$n = 0 : 0.1 : 5;$

$c = \frac{(n.^3)}{2} - n.^2 + n;$

$c_1 = \frac{(n.^3)}{3} - n.^2 + n;$

$c_2 = \frac{(n.^3)}{4} - n.^2 + n;$

$c_3 = \frac{(n.^3)}{5} - n.^2 + n;$

$\text{plot}(n, c, n, c_1, n, c_2, n, c_3)$

$xlabel('n*')$

$ylabel('c(n*)')$

1.7 Figure 4.3

$n = 0 : 0.1 : 5;$

$c = \frac{(n.^3)}{6} - \frac{(n.^2)}{3} + n;$

$c_1 = \frac{(n.^3)}{9} - \frac{(n.^2)}{3} + n;$

$c_2 = \frac{(n.^3)}{12} - \frac{(n.^2)}{3} + n;$

$c_3 = \frac{(n.^3)}{15} - \frac{(n.^2)}{3} + n;$

plot$(n, c, n, c_1, n, c_2, n, c_3)$

$xlabel('n*')$

$ylabel('c(n*)')$

1.8 Figure 4.4

$b1 = 0.1; in = 0.01; Final = 5;$

$Fb = \frac{(Final-b1)}{in};$

$n = b1 : in : Final;$

hold on

$c = 0.2;$

for $b = 1 : Fb + 1,$

$k(b) = \frac{n(b)^3}{(n(b)^2-n(b)+c)};$

$e(b) = \frac{1}{n(b)} * (1 - \frac{c}{n(b)});$

end

grid on

plot(k, n)

$xlabel('\mathbf{k, Carrying\ capacity}')$

$ylabel('\mathbf{n, Fish\ population}')$

1.9 Figure 4.5

$b1 = 0.1; in = 0.01; Final = 5;$

$Fb = \frac{(Final-b1)}{in};$

$n = b1 : in : Final;$

hold on

$c = 0.2;$

for $b = 1 : Fb + 1,$

$k(b) = \frac{n(b)^3}{(n(b)^2 - n(b) + c)};$

$e(b) = \frac{1}{n(b)} * (1 - \frac{c}{n(b)});$

end

grid on

plot(k, e)

$xlabel('\mathbf{k, Carrying\ capacity}')$

$ylabel('\mathbf{E, Fishing\ effort}')$

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