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On Certain Spaces of Ideal Operators

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ABSTRACT

We determine some important spaces of ideal operators and ideal characteristics. Special consideration is given to Frechet spaces, Spaces of finite rank operators and spaces of Hahn-Banach extension operators. The characteristics of ideals and related properties in these spaces as well as in some of their dual spaces are obtained.

Keywords: Operator Ideals, Spaces of Ideal

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1 Introduction

The study of Abrahamsen et al[1] on unconditional ideals of finite rank operators gave characterizations of when $F(Y,X)$ is a u-ideal in $W(Y,X)$ for every Banach spaces X and Y in terms of nets of finite rank operators approximating weakly compact operators. Similar characterizations were given for the cases when $F(Y,X)$ is a u-ideal in $W(Y,X)$ for every Banach space Y, when $F(Y,X)$ is a u-ideal in $W(Y,X)$ for every Banach space Y and when $F(Y,X)$ is a u-ideal in $K(Y,X)$ for every Banach space Y. Abrahamnsen et al[2] defined and studied λ-strict ideals in Banach spaces in which for $\lambda = 1$ means strict ideals. Strict u-ideals in their biduals are known to have the unique ideal property and the study in [2] revealed that the λ-strict u-ideals also have unique properties in their biduals, at least for $\lambda > 1/2$.

Lima et al [13] on the Geometry of operator spaces considered bounded approximation properties via nuclear and integral operators. Starting with a Banach space X and a Banach operator A, they determined the λ bounded approximation property for A ($\lambda - BAP$ for A) and showed that for every Banach space Y and every Operator $T \in A(X,Y)$, there exists a net $(S\alpha)$ of finite rank operators on X such that $S\alpha \to I_X$ uniformly on compact subsets of X and $\limsup \|TS\alpha\|_{\text{A}} \leq \lambda \|T\|_{\text{A}}$. They further proved that the weak λ-BAP is precisely the λ-BAP for the ideal N of nuclear operators. Lima [11] conducted a study on the metric approximation properties in Banach spaces where it was shown that if a Banach space Y is a u-ideal in its bidual Y^{**} with respect to the canonical projection on the third dual Y^{***}, then Y^* contains "many" functionals admitting a unique norm-preserving extension to Y^{**} and the dual unit ball B_{Y^*} is the norm-closed convex hull of its weak * strongly exposed points. Consequently, Martsinkevits and Poldvere[17] in their study on the structure of the dual unit ball of strict $u-$ ideals showed that if Y is a strict u-ideal in a Banach space X with respect to an ideal projection P on

X^*, and X/Y is separable, then $B_{Y^*}(X)$ is the τP closed convex hull of functionals admitting a unique norm-preserving extension to X, where τP is a certain weak topology on Y^* defined by the ideal projection P. A question that arises still is: if X is a Banach space which is a strict u-ideal in it bidual and Y any separable subspace of X, then is Y a strict u-ideal in its bidual?, and is X separably determined?. Our study provides a partial solution to this question.

Lima et al[9] developed a Compact Approximation Theory where they showed that a Banach spaces X has the compact approximation property if and only if for every Banach space Y and every weakly compact operator $\mathrm{T} : \mathrm{Y} \to X$, the space $\mathfrak{S} = \{S \circ T : S$ is a compact operator on X $\}$ is an ideal in $\mathfrak{J} = \mathrm{span}(\mathfrak{S}, \{T\})$ if and only if for every Banach space Y and every weakly compact operator $\mathrm{T} : \mathrm{Y} \to X$, there is a net (S_γ) of compact operators on X such that $\sup_\gamma \|S_\gamma T\| \leq \|T\|$ in the strong operator topology. Similar results for dual spaces were also shown. Now, let $X \subseteq Y$ be Banach spaces and let $\mathrm{A} \subseteq \mathrm{B}$ be closed operator ideals. Let Z be a Banach space having the Radon-Nikod'ym property. Lima, and Oja[13] showed that if $\Phi : \mathrm{A}(Z, X)^* \to \mathrm{B}(Z, Y)^*$ is a Hahn-Banach extension operator, then there exists a set of Hahn-Banach extension operators $\phi_i : X^* \to Y^* i$ $i \in I$, such that $Z = \sum_{i \in I} \oplus_i Z_{\Phi\phi_i}$, where $Z_{\Phi\phi_i} = \{z \in Z : \Phi(x^* \otimes z) = (\phi_i x^*) \otimes z, x^* \in X^*\}$. Further if $\mathrm{B}(\mathrm{Z}, \hat{Y})$ is an ideal in $\mathrm{B}(\widehat{Z}, Y)$ for all equivalently renormed versions $\widehat{Z}$ of Z, then there exist Hahn-Banach extension operators $\Phi : \mathrm{A}(Z, X)^* \to \mathrm{B}(Z, Y)^*$ and $\Phi : X^* \to Y^*$ such that $Z = Z_{\Phi\phi}$.

Hamard and Lima [7] investigated Banach spaces X such that X is an M-ideal in X^{**}. Subspaces, quotients and c_0-sums of spaces which are M-ideals in their biduals are again of this type. A non-reflexive space X which is an M-ideal in X^{**} contains a copy of c_0. In their study, they showed that if $K(X)$ is an M-ideal in $L(X)$ then X is an M-ideal in X^{**}. Also, if X is reflexive and $K(X)$ is an M-ideal in $L(X)$, then $K(X)^{**}$ is isometric to $L(X)$, that is, $K(X)$ is an M-ideal in its bidual. Moreover, for real such spaces, $K(X)$ contains a proper M-ideal if and only if X or X^* contains a proper M-ideal. The proofs of these results are based upon the fact that X is an M-ideal in X^{**} if and only if the natural projection from X^{***} onto X^* is an L-projection. Using local reflexivity it is shown that if X is an M-ideal in X^{**} and X is non-reflexive, then X contains almost isometric copies of c_0. From this it follows that subspaces and quotients are isomorphic to dual spaces are reflexive.

Lima [10] studied strict u-ideals in Banach spaces. A Banach space X is a strict u-ideal in its bidual when the canonical decomposition $X^{***} = X^* \oplus X^\perp$ is unconditional. In characterizing Banach spaces which are strict u-ideals in their bidual it is shown that if X is a strict u-ideal in a Banach space Y then X contains c_0. It is also shown that ℓ_∞ is not a u-ideal. Let X be a subspace of a Banach space Y, X is said to be a summand of Y if it is the range of a contractive projection and that X is an ideal in Y if $X^\perp$ is the kernel of a contractive projection on Y^*.

A norm one operator $\phi : X^* \to Y^*$ such that $\phi(x^*)(x) = x^*(x)$ is said to be a Hahn-Banach extension operator. The set of all such ϕ is denoted by $\mathrm{B}(X, Y)$. For every $\phi \in \mathrm{B}(X, Y)$ we have

$$Y^* = X^\perp \oplus \phi(X^*)$$

Let i_X be the natural embedding $i_X : X \to Y$. $P_\phi = \phi \circ i_X^*$ is a norm one projection on Y^* with $\ker P = X^\perp$. X is an ideal in Y if and only if $\mathrm{B}(X, Y) \neq \emptyset$ (see [9 If $\|x^\perp + \phi(x^*)\| = \|x^\perp - \phi(x^*)\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$ then X is a u-ideal in Y and that ϕ is unconditional. ϕ is unconditional if and only if $\|I - 2P_\phi\| = 1$ which gives a well-known notion of an M-ideal [5, 8] if $\|x^\perp + \phi(x^*)\| = \|x^\perp\| + \|\phi(x^*)\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$.

The operator space structures and the the Algebra of Ideals/Modules has progressively impacted on various findings. Some more elaborate details regarding this subject can be found in among other references ([16][14][21][19] [4]) and most recently [20]. Mathews[16] investigated algebraic questions about the structure of $B(E)$ and ideals thereof, where $B(E)$ is the Banach algebra of all operators on a Banach space E. The study showed that there exist many examples of reflexive Banach spaces E such that $B(E)$ is not Arens regular. In the Banach Algebra setting, the study determined classes of modules in a C^*- algebra that admit the Arens's product. The approximate properties, the nuclear operators and integral properties of such modules were determined. Linus[14] studied the Ideals and Boundaries in Algebras of Holomorphic Functions. In particular, the study investigated the spectrum of certain Banach algebras. Properties like generators of maximal ideals and generalized Shilov boundaries are studied. In particular it was shown that if the $\delta-$equation has solutions in the algebra of bounded functions or continuous functions up to the boundary of a domain $D \subset\subset \mathbb{C}^n$ then every maximal ideal over D is generated by the coordinate functions. This implies that the fibres over D in the spectrum are trivial and that the projection on $\mathbb{C}^n$ of the $n-1$ order generalized Shilov boundary is contained in the boundary of D. The complex analytic theory propagated was used in the determination of the generators of functional algebras in manifolds.

The $M-$embedments, ones sided structures, multipliers and related theories of $r-$ ideals and $l-$ideals were developed by Sonia[21]. The main idea here was to enrich the non-commutative attributes and a generalization of ideal structures to specified operator spaces. Recently, Bence[4] developed the clarity of Algebras of operators on Banach spaces, and homomorphisms thereof. The study was devoted to the homomorphisms and perturbations of homomorphisms of such algebras with a keen focus on perturbations of homomorphisms between Banach algebras. Indeed, the Finiteness and stable rank of algebras of operators on Banach spaces were determined. The results of the study showed that it is possible to develop a unified Theory of maps and functors over modules. Using the methods proposed by Bence[4], Saeid[20] characterized the properties of $\lambda-$continuous functions in vector valued topological spaces. This justifies the consistent development of the the Theory of Maps with respect to the hereditary algebras in Frechet spaces. In fact, Rahul's study in [18] on the study of some classes of operator spaces considers two classes of operators on Banach spaces. One is the class of local isometries and the second is the class of projections which are related to isometries. The isometries guarantee the preservations of local angles and distances while projections guarantee the existence of operator ideals and modules in the general setting.

2 Frechet Space of Operator Ideals

The notions in both Hilbert and Banach spaces can be generalized if a Frechet space say F is taken an ambient space. Therefore, this section considers the ideal properties in the Frechet spaces with respect to the approximate identities, density and smoothness. Bounded approximate identity is a key concept in the theory of amenability of algebras. We show that algebra of compact operators on Frechet space X has both the right and left locally bounded approximate identities. Sufficient conditions for the existence of these identities are established based on the geometry properties of the Frechet space X and its dual space X^* respectively.

A topological linear space X is referred to as a *lcs* if it has a local neighbourhood base comprising convex sets. The *lcs* X is referred to as reflexive if it coincides with the continuous dual of its continuous dual space, that is $X = X^*$. A *lcs* is called a metrizable *lcs* if it possesses countable local neighbourhood base. A Frechet space X is a complete, metrizable *lcs*. Its notions therefore generalize Banach space and Hilbert spaces. Any algebra $\mathbb{A}$ equipped with a structure of *lcs* with respect to which the product is separately continuous is a topological algebra[3]. So, a Frechet algebra is a complete topological algebra of which an increasing countable collection $\{p_i;\ i \in \mathbb{N}\}$ of sub-multiplicative continuous semi-norms determines its topology. A Frechet algebra $\mathbb{A}$ is called amenable if given an $\mathbb{A}-$bimodule Y , every continuous derivation from $\mathbb{A}$ to the dual bimodule Y^* is inner.

Given *lcs* X and Y . Let $T : X \to Y$ be a linear operator. Then, $T : X \to Y$ is called bounded if for some neighbourhood U in X, $T(U)$ is bounded in Y and the operator ideal $U(X; Y)$ is closed if $U = \bar{U}$.

A space X is said to have an unconditional partition of the identity (UPI) if for a sequence $\{T_n\}_n$ of continuous linear operators $T_n : X \to X$ we have $dim T_n(X)$ finite and $\sum_i T_i s$, where convergence is unconditional, $s \in X$.

The next results then follow:

Proposition 2.1. *Suppose X^* is the dual of a Frechet space X, there exists a bijection between operators on X^* and the strict inductive limit of the inductive system of continuous linear operators of Banach spaces.*

Proof. Let $i, j \in I$ such that $j \geq i$, we define a map $f_{ij} : X_i \longrightarrow X_j$ such that $U_i \subset U_j$ where U_i and U_j are 0-neighbourhoods in X_i and X_j respectively and f_{ij} is continuous with $\{X_i\}_i$ being family of Banach spaces. Hence, we identify X^* as the strict direct limit of sequence of Banach spaces $\{X_i\}$. That is $X^* = \lim \longrightarrow X_i = \bigcup X_i = X^* \quad (i = 1, 2, \ldots)$ with $f_{ij} \circ f_{jk} = f_{ik}$ satisfied for $j \geq i, k \geq j. X^*$ is endowed with strict inductive limit topology where $f_i : X_i \longrightarrow X^*$ is continuous such that $f_i(s_i) = s^*$ and $f_{ij}(s_i) = s_j$. Hence, X^* is a complete lcs. We identify X^* as the dual of a Frechet space X. Moreover, given $i \in I$. Let $T_i : D(T_i) \subset X_i \longrightarrow X_i$. $\{T_i : i \in I\}$ can be seen as an inductive system of operators in such a way that for $s_i \in D(T_i) \subset X_i$ and $i > j$.

$$T_i(f_{ji}(s_j)) = f_{ji}(T_j(s_j)).$$

We then define T^* as the inductive limit of the inductive system $\{D(T_i) : i \in I\}$ using $T^*(s^*) = f_i(T_i(s_i))$ or $f_i^{-1}(T^*(s^*)) = T_i\left(f_i^{-1}(s^*)\right)$ where $s^* \in D(X^*)$ with $i \in I$. Therefore, we refer to T^* as the direct limit of $\{T_i : i \in I\}$. We have that T^* is a linear operator. Hence, for each $i, T_i \in L(X_i)$, there exists $T^* \in L_I(X^*)$. In the sequel, we finally have the following relation. $T \in L_I(\lim_{\longleftarrow} X_i) = L_I(X)$ and T^* $L_I(\lim_{\longrightarrow} X_i) = L_I(X^*)$.

□

Proposition 2.2. *Suppose X and Y are Frechet spaces where X_0 and Y_0 are subspaces of X and Y respectively. Let X be quasi normable and Y be reflexive. If $R \in L_I(X_0, X) \subseteq L(X_0, X)$ and $S \in M_I(Y, Y_0) \subseteq L(Y, Y_0)$, then the algebra of compact operators $K_I(X, Y)$ is an ideal in $L_I(X, Y)$.*

Proof. Suppose $R \in L_I(X_0, X), S \in M_I(Y, Y_0)$ and $T \in K_I(X, Y)$. We need to show that $K_I(X, Y)$ is an ideal. Since $K_I(X, Y) \subseteq L_I(X, Y)$, it is not empty. By definition, there exists some neighbourhood $U_0 \subset X_0$ and a bounded subset $B \subset X$ such that

$$RU_0 \subset B \tag{2.1}$$

Since X is quasi normable, there are 0-neighbourhoods U and V with $V \subset U$ such that for every $\epsilon > 0$ we have $V \subset B + \epsilon U$. Hence, by definition there exists a compact set $W \subset Y$ where TV is relatively compact in Y. That is

$$TV \subset W \tag{2.2}$$

Lastly, since Y is reflexive, the relatively compact set TV is a bounded set in Y. Hence, there exists by definition a compact set $G \subset Y_0$ where $S(TV)$ is relatively compact in Y_0. That is

$$S(TV) \subset G \tag{2.3}$$

From relations (4.1) and (4.2), $RU_0 \subset B + \epsilon U$. Hence,

$$T(RU_0) \subset W \tag{2.4}$$

From relations (4.3) and (4.4), since $T(RU_0)$ is relatively compact, which implies that it is bounded in a reflexive Frechet space Y. Hence,

$$ST(RU_0) \subset G.$$

This implies that $STR \in K_I(X_0, Y_0)$. Therefore, $K_I(X, Y)$ is an ideal.

□

Proposition 2.3. *Suppose X is a Frechet space. An UPI for X implies an UPI for X^*.*

Proof. Let a Frechet space X has UPI. That is, for a continuous linear sequence of operators $\{T\}_i \subset L_I(X)$ with $\dim(T_i(s)) < \infty$ and $i \in \mathbb{N}$, we have $\sum_i^n T_i(s) = s$. Let s_i converge to s weakly in X. Then, we have

$$\sum_i T_i(s_i - s) = \sum_i (T_i s_i - T_i s) \longrightarrow 0.$$

Therefore, for all k there exists j and $c > 0$ such that

$$\sum_i p_k(T_i(s_i - s)) \leq c p_j(s_i - s)$$

Therefore,

$$\begin{aligned} |c p_j(s_i) - c p_j(s)| &\geq \left| \sum_i p_k(T_i(s_i)) - \sum_i p_k(T_i(s)) \right| \\ &\leq \sum_i p_k(T_i(s_i - s)) \leq c p_j(s_i - s), \end{aligned}$$

Hence,

$$\left|\sum_i p_k\left(T_i\left(s_i\right)\right)-\sum_i p_k\left(T_i(s)\right)\right| \leq c p_j\left(s_i-s\right).$$

Then for $\lambda \in X^*, s \in X$ we have

$$\begin{aligned}
\left|c p_k^*(\lambda) p_j\left(s_i\right)\right|-\left|c p_k^*(\lambda) p_j(s)\right| \geq \sum_i \mid \lambda\left(T_i\left(s_i\right)\right|-\sum_i\left|\lambda\left(T_i(s)\right| \leq \sum_i\right| \lambda\left(T_i\left(s_i-s\right)\right) \\
\leq c p_k^*(\lambda) p_j\left(s_i-s\right)
\end{aligned}$$

That is

$$\sum_i\left|\lambda\left(T_i\left(s_i\right)\right)\right|-\left|\lambda \sum_i T_i(s)\right| \leq c p_k^*(\lambda) p_j\left(s_i-s\right).$$

Since the summation is over i and also since X has UPI, i.e. $\sum_i T_i s=s$, we then have

$$\sum_i \mid \lambda\left(T_i\left(s_i\right)\right) \mid-\left|\lambda \sum_i T_i(s)\right| \leq c p_k^{\prime}(\lambda) p_j\left(s_i-s\right)$$

then,

$$\sum_i\left|\lambda\left(T_i\left(s_i\right)\right)\right|-|\lambda(s)| \leq c p_k^*(\lambda) p_j\left(s_i-s\right) \tag{2.5}$$

Let define an operator $T_i^*: X^* \rightarrow X^*$ such that $T_i^{\prime}\left(\lambda s_i\right)=\lambda\left(T_i\left(s_i\right)\right)$ where $\lambda \in X^*$. (4.5) now becomes

$$\begin{aligned}
& \sum_i\left|T_i^*\left(\lambda s_i\right)\right|-|\lambda(s)| \leq c p_k^*(\lambda) p_j\left(s_i-s\right) \\
\Longrightarrow & \left|c p_k^*(\lambda) p_j\left(s_i\right)\right|-\left|c p_k^*(\lambda) p_j(s)\right| \geq \sum_i\left|T_i^*\left(\lambda s_i\right)\right|-|\lambda(s)| \\
& \leq\left|\sum_i T_i^*\left(\lambda s_i\right)-\lambda(s)\right| \leq c p_k^*(\lambda) p_j\left(s_i-s\right).
\end{aligned}$$

This implies that

$$\left|\sum_i T_i^*\left(\lambda s_i\right)-\lambda(s)\right| \leq c p_k^*(\lambda) p_j\left(s_i\right)$$

Let $\epsilon=\max \left(m, c p_k^*(\lambda) p_j\left(s_i\right)\right)$ for $m>0$

$$\therefore\left|\sum_i T_i^*\left(\lambda s_i\right)-\lambda(s)\right| \leq \epsilon$$

Hence, it implies that

$$\sum_i T_i^*\left(\lambda s_i\right)-\lambda s \longrightarrow 0$$

For $s \in X$, we have

$$\sum_i T_i^*(\lambda s)-\lambda s=0$$

that is,

$$\sum_i T_i^*(\lambda s)=\lambda s \text { for } \lambda(s) \in X^*.$$

Therefore X^* has an UPI.

□

Next, we restrict our attention to the separable Banach space setting and investigate certain approximations.

Let X be a separable Banach space. We say that a sequence of compact operators $K_n : X \to X$ is a compact approximating sequence if $\lim_{n\to\infty} K_n x = x$ for every $x \in X$. So $(K_n), n \in \mathbb{N}$ is an approximating sequence if each K_n is finite-rank operator. We extend the notion in[10], we say that a Banach space X has (UKAP) if there is a compact approximating sequence $K_n : X \to X$ such that $\lim_{n\to\infty} \|I - 2K_n\| = 1$. For the complex case of a complex Banach space we say that X has a complex (UKAP) if there is a compact approximating sequence such that $\lim_{n\to\infty} \|I - (1+\lambda)K_n\| = 1$ whenever $|\lambda| = 1$. This condition imposes the ideal property.

Lemma 2.1. *[6]*

- *(i) Let X be a separable Banach space.Then X has $(UKAP)$ if and only if for every $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that for every $x \in X$ and every n and every $\theta_j = \pm 1, 1 \le j \le n$, we have $\sum_{n=1}^{\infty} A_n x = x$ and $\left\|\sum_{j=1}^{n} \theta_j A_j x\right\| \le (1+\varepsilon)\|x\|$.*
- *(ii) Let X be a separable complex Banach space. Then X has complex (UKAP) if and only if for every $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that for every $x \in X$ and every n and every $|\theta_j| \le 1$, for $1 \le j \le n$, we have $\sum_{n=1}^{\infty} A_n x = x$ and $\left\|\sum_{j=1}^{n} \theta_j A_j x\right\| \le (1+\varepsilon)\|x\|$.*

Proposition 2.4. *Let X be a separable Banach space. If X has (UKAP) then X is a $u-$ ideal and $K(X)$ is a $u-$ ideal in $L(X)$.*

Proof. The fact that X is a $u-$ ideal follows easily from the previous chapter. The remainder conclusion also follows since if z is a finite-dimensional subspace of $K(X)$ and $\varepsilon > 0$ We can find K compact such that $\|KS - S\| \le \epsilon\|S\|$ for $S \in Z$ and $\|1 - 2K\| \le 1 + \epsilon$. Then consider $\Lambda(S) = KS$ for $S \in L(X)$. ☐

In the Theorems that follows we show the converse of the above proposition under the condition of separability and reflexivity.

Theorem 2.2. *Let X be separable reflexive Banach space. Then X has UKAP if and only if $K(X)$ is a $u-$ ideal in $L(X)$.*

Proof. We denote by $P : L(X)^* \to L(X)^*$ the projection with $kerP = L(X)^{\perp}$ and by $T : L(X)^{\cdot} \to K(X)^{**}$ the induced operator. For $x \in X$ and $x^* \in X^*$, we let $x \otimes x^* \in K(X)^*$ be the linear functional given by $\langle K, x \otimes x^* \rangle = \langle Kx, x^* \rangle$. This functional has a natural extension to $L(X)$, also denoted by $x \otimes x^*$. Let $u \in X$ and $v \in X^*$ be points of Frechet smoothness with $\|u\| = \|v^*\| = 1$. We suppose $u^* \in S_{X^*}$ and $v^{\cdots} \in S_X$ satisfy $u^*(u) = v^*(v) = 1$. Let $A : X \to X$ be the rank-one operator given by $Ax = v^*(x)u$. Then A is a point of Frechet smoothness in $L(X)$. For real λ we have $\|A + \lambda I\| = 1 + \lambda u^*(v) + \alpha(|\lambda|)$. Hence in $K(X)^{**}$ we obtain $\|A + \lambda T(I)\| \le 1 + \lambda u^*(v) + \alpha(|\lambda|)$ in particular, $\langle v \otimes u^*, A + \lambda T(I)\rangle \le 1 + \lambda u^*(v) + \alpha(|\lambda|)$. Hence $\langle v \otimes u^*, T(I)\rangle = u^*(v)$. Now since the points of Frechet smoothness form a dense G_δ in both X and X^* we have for every $\in X, x^* \in X^*, \langle x \otimes x^*, T(I)\rangle = x^*(x)$.

Now by lemma 2.2 in [5], there is a net (K_d) in $K(X)$ such that K_d converges $weak^*$ to $T(I)$ and limsup $\|I - 2K_d\| = 1$. But then $K_d \to I$ for the weak operator topology. Hence for each d we can find $L_d \in \text{co}\{K_e : e \ge d\}$ such that $K_d \to I$ for the strong operator topology. It follows that there is a compact approximating sequence (M_n) in $K(X)$ such that $\lim \|I - 2M_n\| = 1$. ☐

Theorem 2.3. *Let X be a complex Banach space such that X^* is separable. Then X has complex (UKAP) if and only if $K(X)$ is an h-ideal in $L(X)$ and X is an h-ideal.*

Proof. For $x^* \in X^*$ and $x^{**} \in X^{**}$ we use $x^* \otimes x^{**}$ to denote the element of $K(X)^*$ given by $\langle S, x \otimes x^* \rangle = x^{**}(S^* x^*)$. It is easy to note that the formula then defines $x^* \otimes x^{**} \in L(X)^*$. We define $L(X) \to K(X)^{**}$. By this it is possible to define an operator $H : X^{**} \to X^{**}$ such that $\langle x \otimes x^*, T(I)\rangle = \langle x^*, Hx^{**}\rangle$, we first argue that H is Hermitian. In fact suppose $|\lambda| = 1$. It follows from the fact that P is Hermitian that $\|1 - (1+\lambda)H\| \le 1$. Thus if ϕ is a state on $L(X)^{**}$ then $|1 - (1+\lambda)\phi(H)| \le 1$. Hence $\phi(H)$ is real and further $0 \le \phi(H) \le 1$. Hence H is a hermitian.
Next we argue that there is an Hermitian $H_0 : X \to X$ such that $H = {H_0}^{**}$. In fact for any real $t, e^{(itH)}$ is an isometric isomorphism on X^{**}. We recall that X is necessarily a strict $h-$ ideal by Theorem 6.6 in [6]. Applying theorem 5.7 in [6]

we can deduce that $e^{(itH)}$ maps X to X and it is weak * continuous on differentiating we conclude that $H = H_0{}^{**}$ where $H_0 : X \to X$ is the restriction of H.

Next we recall that the collection of points of Frechet smoothness in X form a dense G_δ as do the points of Gateaux smoothness in X^*. Let us suppose that $u \in S_X$ is a point of Frechet smoothness and that $u^* \in S_{X^*}$ is a point of Gateaux smoothness and v^{**} is the corresponding exposed functional in $S_{X^{**}}$. We define the rank-one operator $Ax = v^*(x)u$ and claim that

$$\|A + \xi I\| = 1 + \mathcal{R}\xi\left(v^{**}\left(u^*\right)\right) + O(|\xi|).$$

In fact

$$\|A + \xi I\| \geq \mathcal{R}v^{**}\left(A^* u^* + \xi u^*\right) = 1 + \mathcal{R}\left(\xi v^{**}\left(u^*\right)\right).$$

Conversely, for any ξ we may pick

$$x^*(\xi) \in S_{X^*} \text{so that} 0 \leq x^*(\xi)(u) \leq 1 \text{and} \left\|\left(A^* + \xi I\right)\left(x^*(\xi)\right)\right\| \geq \|A + \xi I\| - |\xi|^2.$$

Letting $\xi \to 0$ we observe that if x^* is any weak* cluster point then $(0 \leq x^*(u)) \leq 1$ and $\|Ax^*\| = 1$. Hence $\lim \xi \to 0 \text{as} x^*(\xi) = u^*$ weak*. However, this implies, since u is a point of Frechet smoothness, that $\lim \xi \to 0\, \|x^*(\xi) - u^*\| = 0$. It now follows immediately from the Gateaux smoothness of the norm at v^* that

$$\|A + \xi I\| = 1 + \mathcal{R}\left(\xi v^{**}\left(u^*\right)\right) + O(|\xi|).$$

Using the formal identity

$$\|T(A) + \xi T(I)\| \leq 1 + \mathcal{R}\left(\xi v^{**}\left(u^*\right)\right) + O(|\xi|)$$

and hence

$$\mathcal{R}\left\langle\left(A^{**} + \xi H\right) v^{**}, u^*\right\rangle \leq 1 + \mathcal{R}\left\langle\xi v^{**}, u^*\right\rangle + O(|\xi|).$$

It follows that $\langle Hv^{**}, u^*\rangle = \langle v^{**}, u^*\rangle$. If we fix u^*, the collection of all v^{**} as v^* ranges all over all points of Gateaux smoothness spans a weak *-dense subspace of X^* and so $H_0 = I$. Now there is a net (K_d) in $K(X)$ such that K_d converges weak * to $T(I)$ and limsup $\|I - (1+\lambda)K_d\| = 1$ whenever $|\lambda| = 1$. □

3 Locally Uniform Rotundity

The properties of locally uniform rotund norms are useful in showing that sufficiently many simple tensors for example $x^* \otimes y^{**}$ when viewed as functionals on the finite rank operator $F(X, Y)$ have unique Hahn-Banach extensions to the space of all bounded operators $K(X, Y)$.

Some of the basic facts on locally uniform rotund norms useful in the sequel are:

Definition 3.1. The norm on a Banach space Y is locally uniformly rotund at a point $y \neq 0$ if $\lim_n \|y - y_n\| = 0$ whenever $(y_n) \subseteq Y$ with $\|y_n\| = \|y\|$ for all n and $\lim_n \left\|\frac{y+y_n}{2}\right\| = \|y\|$. The norm is locally uniformly rotund if it is locally uniformly rotund at every point $y \neq 0$ in Y.

The result that follows provides the link between the denting points and the norm rotund characteristic:

Proposition 3.1. *Let Y be a Banach space and let $y \in Y \backslash \{0\}$. The following are equivalent:*

- *(a) The norm $\|.\|$ is locally uniformly rotund at y.*
- *(b) If $(y_n) \subseteq Y$ is such that* $\lim_n \left(2\|y\|^2 + 2\left\|y_n\right\|^2 - \left\|y + y_n\right\|^2 = 0\right)$, *then* $\lim_n \|y - y_n\| = 0$.
- *(c) The function $\delta_y : [0, 2\|y\|] \to [0, \|y\|]$,defined by the formula $\delta_y(\varepsilon) = \inf\{\|y\| + \|u\| - \|y+u\| : \|u\| = \|y\|, \|y-u\| \geq \varepsilon\}$,satisfies $\delta_y(\varepsilon) > 0$ whenever $0 < \varepsilon \leq 2\|y\|$.*

We provide the following renorming result whose proof can be given using the above proposition.

Lemma 3.1. *Let Y be a Banach space with $\|.\|$ and let $y \in Y\backslash\{0\}$. Assume $\|. \mid \|$ is an equivalent norm on Y such that $\|.\| \mid$ is locally uniformly rotund at y. Let $a, b > 0$ and define a new norm $|.|$ on Y by $|y|^2 = a\|y\|^2 + \||y|\|^2$. Then $|.|$ is locally uniformly rotund at y.*

Remark 3.1. From the lemma above if we choose a close to 1 and b close to 0 , we may assume that the new norm is "close" to the original norm. Precisely this leads to the next lemma.

Lemma 3.2. *Let Y be a Banach space, let $Z \subseteq$ Y be a closed separable subspace and let $\varepsilon > 0$ be given. There exists an equivalent norm $\||.|\|$ on Y such that $B_Y(0,1) \subseteq B_{(Y, \||. \mid \|)}(0,1) \subseteq B_Y(0,1+\varepsilon)$ and such that the norm $\|..\|$ is locally uniformly rotund at every point $Z \neq 0$ in Z.*

Proof. Z is a separable space and so by theorem 11.2.6 in [12]. It has an equivalent locally uniform rotund norm can be extended to an equivalent norm on Y in such a way that this new norm is locally uniformly rotund at every $z \neq 0$ in Z.

Let $\|1\|$ be this equivalent norm on Y which is locally uniformly rotund at every point $z \in Z$, $z \neq 0$. For some $c \geq 1$, we have $\frac{1}{c}\|y\| \leq \||y|\| \leq c\|y\|$ for all $y \in Y$. If $1/c^2 > \theta > 0$ and we let $|y|_\theta^2 = \left(1-\theta c^2\right)\|y\|^2 + \theta\||y|\|^2$, then $|\cdot|_\theta$ is an equivalent norm on Y. $|\cdot|_\theta$ is locally uniformly rotund at every $z \neq 0$ in Z. Let $\varepsilon > 0$ be given, choose θ_0 so small that $\sqrt{1-\theta_0 c^2 + \theta_0/_{c^2}} \geq (1+\varepsilon)^{-1}$. Finally, we redefine $\|.\| = \|.\|_\theta$. and get $(B_{(Y,\|\cdot\|)}(0,1) \subseteq B_{(Y}, \||\cdot|\|)(0,1) \subseteq B_{(Y,\|\cdot\|)}(0,1+\varepsilon)$ as desired.

□

A similar result as the one above for subspaces of dual spaces will be of interest to us too. We shall consider finite-dimensional subspaces. This is considered in the Lemma that follows.

Lemma 3.3. *Let Y be a Banach space, let $F \subseteq Y^*$ be finite-dimensional subspace and let $\varepsilon > 0$. There exists an equivalent norm $\|.\|$ on Y such that*

$$(B_Y(0,1) \subseteq B_{(Y}, \||.|)(0,1) \subseteq B_Y(0,1+\varepsilon)$$

and such that the dual norm of $\||.|\|$ on Y^* is locally uniformly rotund at every point $y^* \neq 0$ in F

Proof. Let $\theta > 0$ be given. Since $\dim F < \infty$, there exists an equivalent locally uniformly rotund norm $|.|$ on F. Moreover we may assume

$$B_F(0,1) \subseteq B_{(F,|,|)}(0,1) \subseteq B_F(0,1+\theta) \cdot \text{Let } B = \operatorname{Conv}\left(B_{Y^*}, B_{(F,|\cdot|)}\right)$$

and let $|.|$ be the norm on Y^* defined by B. B is weak *-compact, so $|.|$ is a dual norm. Moreover,

$$|y^*| \leq \|y^*\| \leq (1+\theta)\,|y^*| \text{ for all } y^* \in Y^* \text{ or } B_{Y^*}(0,1) \subseteq B_{(Y^*,|\cdot|)}(0,1) \subseteq B_{Y^*}(0,1+\theta)\cdot$$

Next we define a new norm on Y^* by $\||y^*|\|_\theta^2 = |y^*|^2 + \theta d^2(y^*, F) \cdot d(y^*, F)$ is computed in the $|.|$ - norm on $Y^* \cdot (Y^*, \||.|\|_\theta)$ is locally uniformly rotund at every point $y^* \neq 0$ in F. We shall show that $\|\,|.|\|_\theta$ is a dual norm. Assume $y_\alpha^* \rightarrow y^*$ weak* with $\||y_\alpha^*|\|_\theta \leq 1$. Choose $(f_\alpha^*) \subseteq F$ such that $|y_\alpha^* - f_\alpha^*| = d(y_\alpha^*, F)$ (f_α^*) is a bounded net and $\dim F < \infty$, so by passing to a subnet, we may assume ${f^*}_\alpha \rightarrow f^* \in F$ in norm. We get

$$d(y_{\alpha^*}^*, F) \leq |y^* - f^*| \leq \liminf_\alpha |\, y_\alpha^* - f_\alpha^* \,| = \liminf_\alpha d(y_\alpha^*, F).$$

Hence,

$$\||y^*|\|_\theta^2 = |y^*|^2 + \theta d^2(y^*, F) \leq \liminf_\alpha |y_\alpha^*|^2 + \theta \liminf_\alpha d^2(y_\alpha^*, F) \leq \liminf_\alpha \||y_\alpha^*|\|_\theta^2.$$

This shows that $\|. \mid \|_\theta$ is a dual norm. Now since
$d(y_\alpha^*, F) \leq |y^*|$, we get $|y^*| \leq ||y^*||_\theta \leq (1+\theta)^{1/2}|y^*|$ Thus

$$(1+\theta)^{-1}\|y^*\| \leq \||y^*|\|_\theta \leq (1+\theta)^{1/2}\|y^*\|.$$

This implies that

$$(1+\theta)^{-1/2}\|y\| \leq \||y^*|\|_\theta \leq (1+\theta)^{-1}\|y\| \text{ for all } y \in Y$$

Let $\varepsilon > 0$. If we use $\||.|\| = (1+\theta)^{-1}\||.|\|_\theta$ as the new norm, and choose θ such that $(1+\theta)^{3/2} \leq 1+\varepsilon$, then $(1+\theta)^{-3/2}\|y\| \leq \||y|\|_{-1} \leq \|y\|$ for all $y \in Y$ and thus

$$B_Y(0,1) \subseteq B_{(Y}, \||\cdot|\|)(0,1) \subseteq B_Y(0,1+\varepsilon)$$

□

Lemma 3.4. *[11] Let X and Y be Banach spaces.Let $x^* \otimes y \in \mathcal{F}(Y,X)^*$ with $x^* \in X^*$ and $y \in Y$.If the norm of Y is locally uniformly rotund at y, then the only Hahn-Banach extension of $x^* \otimes y$ to $\mathcal{H}(Y,X)$ is the trivial one, that is $HB\left(x^* \otimes y\right) = \{x^* \otimes y\}$.*

We shall also need elements in the bidual and the lemma that follows takes care of that.

Lemma 3.5. *[13] Let X and Y be Banach spaces. Let $y^* \otimes x^{**} \in \mathcal{F}(X,Y)^*$ with $y^* \in Y^*$ and $x^{**} \in X^{**}$. If the norm of Y^* is locally uniformly rotund at y^* then the only Hahn-Banach extension of $y^* \otimes x^{**}$ to $\mathcal{H}(X,Y)$ is the trivial one, that is $HB\left(y^* \otimes x^{**}\right) = \{y^* \otimes x^{**}\}$.*

4 Renorming and the Hahn-Banach Extension Operators

Let X and Y be Banach spaces. We let $\mathcal{A}$ and $\mathcal{B}$ denote closed operator ideals. This will ensure that $\mathcal{F}(Y,X) \subseteq \mathcal{A}(Y,X)$ and that $\mathcal{A}(Y,X)$ is a closed subspace of $\mathcal{H}(Y,X)$.

Theorem 4.1. *Let X be a Banach space and Z be a separable subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(\widehat{Y},X)$ is an ideal in $\mathcal{B}(\widehat{Y},X)$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(Y,X),\mathcal{B}(Y,X))$ such that $\left(\psi\left(x^* \otimes z\right)\right)(T) = \left(x^* \otimes z\right)(T)$ for all $x^* \in X^*, z \in Z$, and $T \in \mathcal{B}(Y,X)$*

Proof. For every $\varepsilon > 0$ we can find an equivalent norm $\|\varepsilon\|$ on Y such that $Y_s = (Y, \|.\|_s)$ is locally uniformly rotund at every $z \neq 0$ in Z_ε and such that $B_Y(0,1) \subseteq B_{Y_\varepsilon}(0,1) \subseteq B_Y(0,1+\varepsilon)$. By assumption, $\mathcal{A}\left(Y_\varepsilon, X\right)$ is an ideal in $\mathcal{B}\left(Y_\varepsilon, X\right)$ so there exists a Hahn-Banach extension $Q_\varepsilon : \mathcal{A}\left(Y_\varepsilon, X\right)^* \to \mathcal{B}\left(Y_\varepsilon, X\right)^*$. Now, $\left(Q_\varepsilon\left(x^* \otimes z\right)\right)(T) = \left(x^* \otimes z\right)(T)$ for all $x^* \in X^*$, all $z \in Z$, and all $T \in \mathcal{B}\left(Y_\varepsilon, X\right)$. Let $I_s : Y_s \to Y$ denote the identity mapping. Then $\left\|I_\varepsilon^{-1}\right\| = 1$ and $\|I_\varepsilon\| \to 1$ as $\varepsilon \to 0$. Define $\psi_s \in \mathcal{H}\left(\mathcal{A}(Y,X)^*, \mathcal{B}(Y,X)^*\right)$ by

$$\left(\psi_s(\phi)\right)(T) = \left(Q_s\right)\left(\phi_s\right)\left(T \circ I_s\right), \phi \in \mathcal{A}(Y,X)^*, T \in \mathcal{B}(Y,X)$$

where $\phi_s \in \mathcal{A}\left(Y_s, X\right)^*$ is defined by

$$\phi_\varepsilon(S) = \phi\left(S \circ I_\varepsilon^{-1}\right), \quad S \in \mathcal{A}(Y,X)$$

We can conclude that $\left(\psi_\varepsilon\right), \varepsilon \in (0,1]$ has a subnet converging weak* to some $\psi \in HB(\mathcal{A}(Y,X),\mathcal{B}(Y,X))$ with required property. □

The following Lemma and theorem are crucial in the proof of one of our main results

Lemma 4.2. *[13] Let X be a Banach space and Z be a separable subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. The subset*

$$K_Z = \psi \in H\ B(\mathcal{A}(Y,X),\ \mathcal{B}(Y,X)) : (\psi(x^* \otimes z))(T) = \left(x^* \otimes z\right)(T),\ \forall\ x^*\ \in X^*, z \in Z$$

and $T \in \mathcal{B}(Y,X)\}$ of $\left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi\mathcal{A}(Y,X)^*\right)^*$ is compact in the weak *-topology.

Theorem 4.3. *Let X and Y be Banach spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$.If $\mathcal{A}(\widehat{Y},X)$ is an ideal in $\mathcal{B}(\widehat{Y},X)$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(Y,X),\mathcal{B}(Y,X))$ such that $\left(\psi\left(x^* \otimes y\right)\right)(T) = \left(x^* \otimes y\right)(T)$ for all $x^* \in X^*, y \in Y$, and $T \in \mathcal{B}(Y,X)$.*

Proof. See [11] □

We now state some of our main results:

Proposition 4.1. *If $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{H}(Y,X)$ for every separable Banach space Y, then X has a metric approximation property.*

Proof. Let $L \subseteq X$ be a separable subspace, then we can find a separable ideal Y in X with $L \subseteq Y$. Let $\varphi : Y^* \to X^*$ be a Hahn-Banach extension operator. Let $\psi : \mathcal{F}(Y,X)^* \to \mathcal{H}(Y,X)^*$ be a Hahn-Banach extension operator with $\psi(x^* \otimes y) = x^* \otimes y$ for all $y \in Y$ and $x^* \in X^*$. Let $I : Y \to X$ be the identity map. We see that $\psi^*(I) \in \mathcal{F}(Y,X)^{**}$ there exists a net $(T_\alpha) \subseteq \mathcal{F}(Y,X)$ such that $\sup_\alpha \|T_\alpha\| \leq \|I\| = 1$ and $T_\alpha \to \psi^*(I)$ in the weak *-topology.

In particular

$$\langle T_\alpha y, x^* \rangle = \langle T_\alpha, x^* \otimes y \rangle \to \langle \psi^*(I), x^* \otimes y \rangle = \langle I, \psi(x^* \otimes y) \rangle = \langle Iy, x^* \rangle \text{ for all} y \in Y$$

and $x^* \in X^*$, that is $T_\alpha \to I$ in the weak operator topology. By taking anew net from $\operatorname{Conv}(T_\alpha)$, which we also denote (T_α), we may assume that $T_\alpha \to I$ in the strong operator topology.

Let $\widehat{T}_\alpha = T_\alpha{}^{**\circ}\varphi^*|_X \in \mathcal{F}(X,X)$, then $\left\|\widehat{T}_\alpha\right\| = \|T_\alpha\| \leq 1$ and $\widehat{T}_\alpha$ converges pointwise to the identity I_X on Y. It follows then that X has the metric approximation property. □

Proposition 4.2. *If $\mathcal{F}(\hat{X},X)$ is an ideal in $\mathcal{H}(\hat{X},X)$ for every equivalent renorming $\hat{X}$ of , then X has a metric approximation property.*

Proof. Let $Y = X$. Then there exists $\psi \in HB(\mathcal{F}(X,X), \mathcal{H}(X,X))$ such that $\psi(x^* \otimes x) = x^* \otimes x$ for all $x \in X$ and $x^* \in X^*$. Let $i : \mathcal{F}(X,X) \to \mathcal{H}(X,X)$ be the natural inclusion and define $P = \psi \circ i^*$. Using Theorem 5.4 from [6] with P as the ideal projection we conclude that X has the metric approximation property. Let I_X be the identity operator on X, it can be seen that $\psi^*(I_X) \in \mathcal{F}(X,X)^{**}$. It follows that there exists a net $(T_\alpha) \subseteq \mathcal{F}(X,X)$ such that $\sup_\alpha \|T_\alpha\| \leq= 1$ and $T_a \to I_X$ in the weak *-topology. In particular $\langle T_\alpha x, x^* \rangle \to \langle \psi^*(I_X), x^* \otimes x \rangle = \langle I_X, \psi(x^* \otimes x) \rangle = \langle I_X x, x^* \rangle$ for all $x \in X$ and $x^* \in X^*$, that is $T_\alpha \to I_X$ in the weak operator topology. By taking anew net consisting of convex combinations of the T_α 's we may assume that the net converges in the strong operator topology. □

5 Dual Renorming and the Hahn-Banach Extensions

In this section we proof our major result under dual renorming. We shall replace a separable subspace $Z \subseteq Y$ with a finite-dimensional subspace $F \subseteq Y^*$.

Proposition 5.1. *Let X and Y be Banach spaces, and let F be a finite-dimensional subspace of Y^*.Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X,\hat{Y})$ is an ideal in $\mathcal{B}(X,\hat{Y})$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y))$ such that $(\psi(y^* \otimes x^{**}))(T) = (y^* \otimes x^{**})(T)$ for all $y^* \in F$, all $x^{**} \in X^{**}$ and all $T \in \mathcal{B}(X,Y)$.*

Proof. For all $\varepsilon \in (0,1]$ there exists by an equivalent norm $\|.\|_\varepsilon$ on Y such that the dual norm on Y^* is locally uniformly rotund at every point $y^* \neq 0$ in Y such that

$$B_Y(0,1) \subseteq B_{(Y,\|\|_\varepsilon)}(0,1) \subseteq B_Y(0,1+\varepsilon)$$

Let $Y_\varepsilon = (Y, \|.\|_\varepsilon)$.By assumption $\mathcal{A}(X,Y_\varepsilon)$ is an ideal in $\mathcal{B}(X,Y_\varepsilon)$ so there exists a HahnBanach extension operator $Q_\varepsilon : \mathcal{A}(X,Y_\varepsilon)^* \to \mathcal{B}(X,Y_\varepsilon)^*$.Equivalently, the tensor space gives: $Q_\varepsilon(y^* \otimes x^{**}) = y^* \otimes x^{**}$ for all $y^* \in F_\varepsilon$, and $x^{**} \in X^{**}$. Let $I_\varepsilon : Y_\varepsilon \to Y$ denote the identity mapping. Then $\left\|I_\varepsilon{}^{-1}\right\| = 1$ and $\|I_\varepsilon\| \to 1$ as $\varepsilon \to 0$. Define $\psi_\varepsilon \in \mathcal{H}(\mathcal{A}(X,Y)^*, \mathcal{B}(X,Y)^*)$ by $(\psi_\varepsilon(\phi))(T) = (Q_\varepsilon(\phi_\varepsilon))(I_\varepsilon^{-1\circ}T), \phi \in \mathcal{A}(X,Y)^*, T \in \mathcal{B}(X,Y)$, where $\phi_\varepsilon \in \mathcal{A}(X,Y_s)^*$ is defined by $\phi_s(S) = \phi(I_s{}^\circ S), S \in (X,Y_s)$ such that ψ is constructable. □

Lemma 5.1. *Let X and Y be Banach spaces, and let F be a finite-dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. The subset*

$$K_F = \{\psi \in H\ B(\mathcal{A}(X,Y)\ \mathcal{B}(X,Y)) : (\psi(y^* \otimes x^{**}))(T) = (y^* \otimes) x^{**}(T),$$

for all $y^* \in F, x^{**} \in X^{**}$, and $T \in \mathcal{B}(Y,X)\}$ of $\left(\mathcal{B}(X,Y)\widehat{\otimes}_\pi \mathcal{A}(X,Y)^*\right)^*$ is compact in the weak *-topology.

Theorem 5.2. *Let X and Y be Banach spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X,\hat{Y})$ is an ideal in $\mathcal{B}(X,\hat{Y})$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y))$ such that $(\psi(y^* \otimes x^{**}))(T) = (y^* \otimes x^{**})(T)$ for all $y^* \in Y^*$, all $x^{**} \in X^{**}$ and all $T \in \mathcal{B}(X,Y)$.*

Proof. Let K_F be as defined in Lemma 4.5.1 for every finite-dimensional subspace F of Y^* then each $K_F \neq \Phi$. If $F_1, \ldots\ldots, F_n$ is a finite collection of finite-dimensional subspaces of Y^*, then $\cap_{i=1}^{n} K_{F_i} \neq \Phi$. Let $F = \operatorname{span}(F_1 \cup \ldots . \cup F_n)$. Then F is a finite dimensional subspace of Y^* and $\cap_{i=1}^{n} K_{F_i} \supseteq K_F \neq \Phi$. Now we know by compactness there is a $\psi \in \cap K_F, F \subseteq Y^*, \dim F < \infty$. For all $y^* \in Y^*$ there is a finite-dimensional subspace F of Y^* such that $y^* \in F$. Since $\psi \in K_F$ we have $(\psi(y^* \otimes x^{**}))(T) = (y^* \otimes x^{**})(T)$ for all $x^{**} \in X^{**}$ and $T \in \mathcal{B}(X, Y)$. □

We now state and prove our main result for dual spaces.

Theorem 5.3. *Let X be a Banach space. Given $\mathcal{F}(X, \hat{X})$ is an ideal in $\mathcal{H}(X, \hat{X})$ for every equivalent renorming $\hat{X}$ of X, then X has a shrinking metric approximation property.*

Proof. Starting with a Hahn-Banach extension operator $\psi : \mathcal{F}(X, X)^* \to \mathcal{H}(X, X)^*$ such that $\psi(x^* \otimes x^{**}) = x^* \otimes x^{**}$ for all $x^* \in X^*$ and $x^{**} \in X^{**}$. Theorem 5.2 in [10] now shows that X^* has the metric approximation property. □

6 u-Ideals as Operators and their Duals

Lemma 6.1. *Let X and Y be Banach spaces.Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $T \in \mathcal{B}(Y, X)$. If $\mathcal{A}(Y, X)$ is a u-deals in $\mathcal{B}(Y, X)$ and P is an ideal projection, then there exists a net $(T_\alpha) \subseteq \mathcal{B}(Y, X)$ with limsup $_\alpha \|T - 2T_\alpha\| \leq \|T\|$ such that $y^{**}(T_\alpha{}^* x^*) \to^\alpha (P(x^* \otimes y^{**}))(T)$ for all $x^* \in X^*$ and $y^{**} \in Y^{**}$.*

Theorem 6.2. *Let X be a Banach space and let Z be a separable subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $T \in \mathcal{B}(Y, X)$. If $\mathcal{A}(\hat{Y}, X)$ is a u-ideals in $\mathcal{B}(\hat{Y}, X)$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(Y, X), \mathcal{B}(Y, X)\ )$ such that $(\psi(x^* \otimes z))(T) = (x^* \otimes z)(T)$ for all $x^* \in X^*$ and $z \in Z$ and $T \in \mathcal{B}(Y, X)$. Furthermore the ideal projection $P = \psi \circ i^*$, where i is a natural inclusion, satisfies $\|I - 2P\| = 1$.*

Proof. For every > 0, we can find an equivalent norm $\|.\|_\varepsilon$ on Y such that $Y_\varepsilon = (Y, \|.\|_\varepsilon)$ is locally uniformly rotund at every $z \neq 0$ in Z_ε and such that $B_Y(0,1) \subseteq B_{Y_\varepsilon}(0,1) \subseteq B_Y(0, 1+\varepsilon)$. Let $i_\varepsilon : \mathcal{A}(Y_\varepsilon, X) \to \mathcal{B}(Y_\varepsilon, X)$ be the identity map. By way of assumption $\mathcal{A}(Y_\varepsilon, X)$ is a u-deals in $\mathcal{B}(Y_\varepsilon, X)$

So that there exists an ideal projection $\mathcal{G}_s$ such that $\|I - 2\mathcal{G}_s\| = 1$.We find a Hahn-Banach extension operator Q_ε such that $\mathcal{G}_\varepsilon = Q_\varepsilon \circ i^*$ and $(Q_\varepsilon(x^* \otimes z))(T) = (x^* \otimes z)(T)$ for all $x^* \in X^*$ and $z \in Z_\varepsilon$, and $T \in \mathcal{B}(Y_s, X)$. Let $I_\varepsilon : Y_\varepsilon \to Y$ denote the identity mapping. We define $\phi_\varepsilon \in \mathcal{A}(Y_\varepsilon, X)^*$ using $\phi \in \mathcal{A}(Y, X)^*$,and we define $\psi_\varepsilon \in \mathcal{H}(\mathcal{A}(Y, X)^*, \mathcal{A}(Y, X)^*)$ using Q_ε. Let $\mathcal{S} \in \mathcal{B}(Y, X)^*$ and define $\mathcal{S}^\varepsilon \in \mathcal{B}(Y_\varepsilon, X)^*$ by $\mathcal{S}^\varepsilon(T) = \mathcal{S}(T \circ I_\varepsilon^{-1}), T \in \mathcal{B}(Y_\varepsilon, X)$. For $T \in \mathcal{A}(Y_\varepsilon, X)$ and $\mathcal{S} \in \mathcal{B}(Y, X)^*$ we have $[i^*(\mathcal{S})]_\varepsilon(T) = i^*(\mathcal{S})(T \circ I_\varepsilon^{-1}) = \mathcal{S}(i(T \circ I_\varepsilon^{-1}))$

$$= \mathcal{S}(i_\varepsilon(T) \circ I_\varepsilon^{-1}) = \mathcal{S}^\varepsilon(i_\varepsilon(T))$$
$$= i_\varepsilon^* \mathcal{S}^\varepsilon(T)$$

So that these functionals have the same Q_ε extension. Let $P_\varepsilon = \psi_\varepsilon \circ i^*$. Then we have the following norm estimate:

$$= \sup_{\mathcal{S} \in B_{\mathcal{B}(Y,X)^*}} \quad T \in B_{\mathcal{B}(Y,X)^-} \left|\mathcal{S}(T) - 2\left((Q_\varepsilon)[i^*(\mathcal{S})]_\varepsilon\right)(T \circ I_\varepsilon)\right|$$
$$\leq \delta \in B_{\mathcal{B}(Y,X)^*} \|\mathcal{S}^\varepsilon - 2(Q_\varepsilon \circ i_\varepsilon{}^*)(\delta^\varepsilon)\|(1+\varepsilon)$$
$$\leq \delta \in B_{\mathcal{B}(Y,X)^*} \|I - 2Q_\varepsilon\|(1+\varepsilon) \leq (1+\varepsilon).$$
$$\text{Since } \|I - 2Q_\varepsilon\| \leq 1.$$

Since $(Q_\varepsilon)_{s \in (0,1]} \subseteq \mathcal{H}(\mathcal{A}(Y, X)^*, \mathcal{B}(Y, X)^*) = (\mathcal{B}(Y, X) \widehat{\otimes}_\pi \mathcal{A}(Y, X)^*)^*$ is a bounded net it has a subnet $(Q_{\varepsilon(v)})$ that converges weak * to some Q. In fact Q is a Hahn-Banach extension operator. Next we show that the projection P defined by $P = Q \circ i^*$ is the desired u-ideal projection.
Let $\mathcal{S} \in \mathcal{B}(Y, X)^*$ and $T \in \mathcal{B}(Y, X)$, then $P(\mathcal{S})(T) = Q(i^*(\mathcal{S}))(T) = {}_v^{\lim} Q_{\varepsilon(v)}(i^*(\mathcal{S}))(T)$

$$= \lim{}_v P_{\varepsilon(v)}((\mathcal{S})(T) \text{ so that } \sup_{B(Y,X)} |\ \mathcal{S}(T) - 2P(\mathcal{S})(T)\ |$$

$$= s \in B_{\mathcal{B}(Y,X)^*} \quad T \in B_{\mathcal{B}(Y,X)} \quad \lim_v \mid \mathcal{S}(T) - 2P_{\varepsilon(v)}(\mathcal{S})(T) \mid$$
$$\leq s \in B_{\mathcal{B}(Y,X)^*} \quad T \in B_{\mathcal{B}(Y,X)} \quad \lim_v \|1 - 2P_{\varepsilon(v)}\| \|\mathcal{S}\| \|T\|$$
$$\leq \lim_v \left\|1 - 2P_{\varepsilon(v)}\right\| = 1$$

So we have as required $\|I - 2P\| = 1$. ☐

Lemma 6.3. *Let X be a Banach space and let Z be a separable subspace of a Banach space Y.Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $i : \mathcal{A} \to \mathcal{B}$ be the natural inclusion. The subset*

$$K_z = \left\{ \begin{array}{c} \psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y)) : (\psi\,(x^* \otimes z))\,(T) = (x^* \otimes z)\,(T), \\ \forall x \in X^*, z \in Z \text{ and } T \in \mathcal{B}(Y,X) \text{ and } \|1-2P\| = 1, \text{ where } P = Q \circ i^* \end{array} \right\}$$

is weak-compact in $\left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$.*

Proof. Let $\psi_\varepsilon \subseteq K_z$ be a net which converges weak * to some $\psi \in \left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$. Now we have $(\psi\,(x^* \otimes z))\,(T) = (x^* \otimes z)\,(T)$ for all $x \in X^*, z \in Z$, and $T \in \mathcal{B}(Y,X)$. Let $P = Q \circ i^*$ and thus $\|1 - 2P\| \leq \lim_\alpha \|1 - 2P_\alpha\| = 1$. This shows that K_z is weak $^*-$ closed and since it is a bounded subnet it is weak $^*-$ compact. ☐

Theorem 6.4. *Let X be a Banach space. Given a net $(T_\alpha) \subseteq \mathcal{F}(X,X)$ with $\limsup_\alpha \|1 - 2T_\alpha\| \leq 1$ such that $T_\alpha x \to x$ for all $x \in X$ and $T_\alpha x^* \to x^*$ for all $x^* \in X^*$, then $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{H}(X,Y)$ for every Banach space Y.*

Proof. We shall proof the case for finite rank operators. Let F be a finite dimension subspace of $\mathcal{H}(X,Y)$, and let $\varepsilon > 0$. Let $G = F \cap \mathcal{F}(X,Y)$. Then $K = \overline{U_{T \in B_G} T^*\,(B_{Y^*})}$ is a compact subspace of X^*. By assumption we can find a $T \in \mathcal{F}(X,X)$ such that $\|1-2T\| \leq 1+\varepsilon$ and such that that $\|x^* - T^*x^*\| \leq \varepsilon$ for all $x^* \in K$. We define a linear map $L : F \to \mathcal{F}(X,Y)$ by $L(S) = ST$.Then $\|S - ST\| = \|S^* - T^*S^*\| < \varepsilon\|S\|$ for all $S \in G$ and $\|S - 2L(S)\| \leq (1+\varepsilon)\|S\|$ for all $S \in F$. By applying proposition 3.6 in [4] we find that $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{H}(X,Y)$. ☐

The following results are key to the proof of the second main result under this section.

Theorem 6.5. *Let X and Y be Banach spaces and let F be a finite dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X,\hat{Y})$ is a u-deals in $\mathcal{B}(X,\hat{Y})$ for every equivalent renorming $\widehat{Y}$ of Y, then there exists a*

$$\psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y)) : (\psi\,(y^* \otimes x^{**}))\,(T) = (y^* \otimes x^{**})\,(T),$$

for all $y^* \in F, x^{**} \in X^{**}$, and $T \in \mathcal{B}(X,Y)$ Furthermore, the ideal projection $P = Q \circ i^*$ where i is a natural inclusion, satisfies $\|1 - 2P\| = 1$.

Lemma 6.6. *Let X and Y be Banach spaces and let F be a finite dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$ and let $i : \mathcal{A} \to \mathcal{B}$ be the natural inclusion. The subset*

$$K_Z = \psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y)) : \psi\,(y^* \otimes x^{**})\,(T) = (y^* \otimes x^{**})\,(T), \forall y^* \in F, x^{**} \in X^{**}$$
$$T \in \mathcal{B}(Y,X)\ , \|1 - 2P\| = 1, P = Q \circ i^*$$

is weak *-compact in $\left(\mathcal{B}(Y,X)\widehat{\otimes}_\pi \mathcal{A}(Y,X)^*\right)^*$.

Theorem 6.7. *Let X and Y be Banach spaces and let F be a finite dimensional subspace of Y^*. Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals satisfying $\mathcal{A} \subseteq \mathcal{B}$. If $\mathcal{A}(X,\hat{Y})$ is a u-ideals in $\mathcal{B}(X,\hat{Y})$ for every equivalent renorming $\hat{Y}$ of Y, then there exists a $\psi \in HB(\mathcal{A}(X,Y), \mathcal{B}(X,Y)) : (\psi\,(y^* \otimes x^{**}))\,(T) = (y^* \otimes x^{**})\,(T)$, for all $y^* \in Y^*, x^{**} \in X^{**}$, and $T \in \mathcal{B}(X,Y)$ Furthermore, the ideal projection $P = \psi \circ i^*$ where i is a natural inclusion, satisfies $\|1 - 2P\| = 1$.*

We can now prove our next result under this section.

Proposition 6.1. *Let X be a Banach space. Given $\mathcal{F}(X,\hat{X})$ is a u-ideal in $\mathcal{H}(X,\hat{X})$ for every equivalent renorming $\hat{X}$ of X.Then there is a net $(T_\alpha) \subseteq \mathcal{F}(X,X)$ with $\limsup_\alpha \|1 - 2T_\alpha\| \leq 1$ such that $T_\alpha x \to x$ for all $x \in X$ and $T_\alpha x^* \to x^*$ for all $x^* \in X^*$.*

Proof. We limit ourselves to the case with finite rank operators. By ** above we can find a HahnBanach extension operator

$$\psi : \mathcal{F}(X,X)^* \to \mathcal{H}(X,X)^*$$

such that $(\psi(x^* \otimes x^{**}))(T) = (x^* \otimes x^{**})(T)$ for all $x^* \in X^*, x^{**} \in X^{**}$ and $T \in \mathcal{H}(X,X)$. Let $i : \mathcal{F}(X,X) \to \mathcal{H}(X,X)$ be the natural inclusion, and let $P = \psi \circ i^*$ be the associated u-deal projection. Now there exists a net $(T_\alpha) \subseteq \mathcal{F}(X,X)$ with $\limsup_\alpha \|1 - 2T_\alpha\| \leq 1$ such that $x^{**}(T_\alpha{}^* x^*) \to_\alpha (P(x^* \otimes x^{**}))(T)$ for all $x^* \in X^*$ and $x^{**} \in X^{**}$. Thus for all $x^* \in X^*$ and $x^{**} \in X^{**}$ $x^{**}(T_\alpha{}^* x^*) \to {}_\alpha x^{**}(x^*)$, which means that $T_\alpha{}^* \to I^*$ in the weak operator topology on $\mathcal{H}(X^*, X^*)$. In particular we have $x^*(T_\alpha x) \to {}_\alpha x^*(x)$ for all $x^* \in X^*$ and $x \in X$, so that $T_\alpha \to I$ the weak operator topology on $\mathcal{H}(X,X)$. By choosing a new net in $\operatorname{conv}(T_\alpha)$, still denoted (T_α), we may assume $T_\alpha{}^* \to I^*$ in the strong operator topology on $\mathcal{H}(X^*, X^*)$ and $T_\alpha \to I$ in the strong operator topology on $\mathcal{H}(X,X)$.

□

7 Bounded Approximation Properties through Integral and Nuclear Operators

Let X and Y be Banach spaces. We denote by $L(X,Y)$ the Banach space of all bounded linear operators from X to Y, and by $\mathcal{F}(X,Y)$ and $\mathcal{W}(X,Y)$ its subspaces of finite rank operators and weakly compact operators. Let I_X denote the identity operator on X.

Definition 7.1. We say X has the weak λ - bounded approximation property (weak $\lambda - BAP$) if for every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X and $\limsup_\alpha \|TS_\alpha\| \leq \lambda \|T\|$.

Thus the weak BAP can be characterized as the AP which is bounded for every weakly compact operator. This leads to the following definition

Definition 7.2. Let Let X be a Banach space and let $\mathcal{D} = (\mathcal{D}, \|\|_{\mathcal{D}})$ be a Banach operator ideal. We say that X has the λ - bounded approximation property for $\mathcal{D}$ (weak $\lambda - BAP$ for $\mathcal{D}$) if for every Banach space Y and evry operator $T \in \mathcal{D}(X,Y)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X and $\limsup_\alpha \|TS_\alpha\|_{\mathcal{D}} \leq \lambda \|T\|_{\mathcal{D}}$.

For us to have a complete picture we recall other types of bounded approximation properties involving operator ideals which have been studied.

Definition 7.3. Let $\mathcal{D}$ be an operator ideal. A Banach space X is said to have λ - bounded $\mathcal{D}$ - approximation property (λ - bounded $\mathcal{D} - AP$) if there exists a net $(S_\alpha) \subset \mathcal{D}(X,X)$ With $\sup_\alpha \|S_\alpha\| \leq \lambda$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X.

We establish reformulations of BAP in terms of the boundedness for the Banach operator ideals of strictly integral and integral operators respectively. The following Theorem and lemmas will be key in this direction.

Theorem 7.1. *Let X be a Banach space and I integral operator, and let $1 \leq \lambda < \infty$. The following statements are equivalent.*

(i) *X has the $\lambda - BAP$.*

(ii) $\|T\|_\pi \leq \lambda \|T\|_{\mathrm{I}}$ *for all* $T \in \mathcal{F}(X,X)$.

Lemma 7.2. *Let X be a Banach space, I an integral operator and $1 \leq \lambda < \infty$. If a Banach space Y has the property that for every $T \in \mathrm{I}(X, Y^{**})$ there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ pointwise and $\limsup_\alpha \|TS_\alpha\|_\pi \leq \lambda \|T\|_{\mathrm{I}}$, then every quotient space of Y has the same property.*

Proof. Denote by $q : Y \to Z$ the quotient mapping, and let $U \in \mathrm{I}(X, Z^{**})$. Using well known results about tensor products, $\mathrm{I}(X, Z^{**}) = \left(Z^* \widehat{\otimes}_\varepsilon X\right)^*$ and $Z^* \widehat{\Phi}_\varepsilon X$ is a subspace of $Y^* \widehat{\otimes}_\varepsilon X$, we may consider a norm preserving extension of U. Thus there exists $T \in \left(Y^* \widehat{\otimes}_\varepsilon X\right)^* = \mathrm{I}(X, Y^{**})$, such that $\|T\|_{\mathrm{I}} = \|U\|_{\mathrm{I}}$ and

$$(Ux)(z^*) = \langle U, z^* \otimes x \rangle = \langle T, q^* z^* \otimes x \rangle = (Tx)(q^* z^*) = (q^{**} Tx)(z^*)$$

for all $x \in X$ and $z^* \in Z^*$, meaning that $U = q^{**}T$. Let $S \in \mathcal{F}(X, X)$. Then $US \in \mathcal{F}(X, Z^{**}) = X^* \otimes Z^{**}$ and $\|US\|_\pi = \|q^{**}TS\|_\pi = \|(I_X \otimes q^{**})(TS)\|_\pi \leq \|TS\|_\pi$.

Now if $(S_\alpha) \subset \mathcal{F}(X, X)$ is chosen for T, then we also have $\limsup_\alpha \|US_\alpha\|_\pi \leq \limsup_\alpha \|TS_\alpha\|_\pi \leq \lambda \|T\|_{\mathrm{I}} = \lambda \|U\|_{\mathrm{I}}$ □

Lemma 7.3. *Let X be a Banach space, and let $T \in \mathcal{F}(X, X) = X^* \otimes X$. Then there exists $A \in L(X^*, X^*)$ with $\|A\| = 1$ and $V \in \mathcal{F}(X, X)$ such that $V^* = AT^*$ and $\|T\|_\pi \leq \limsup_\alpha \|j_X V S_\alpha\|_\pi$ for every net $(S_\alpha) \subset \mathcal{F}(X, X)$ converging pointwise to the identity I_X.*

Proof. Let $T = \sum_{n=1}^m x^* \otimes X$. Using the canonical description $\left(X^* \widehat{8}_\pi X\right)^* = L(X^*, X^*)$, we find $A \in L(X^*, X^*)$ with $\|A\| = 1$ such that $\|T\|_\pi = \sum_{n=1}^m (Ax_n{}^*)(x_n) = \mathrm{trace}(V)$, where $V = \sum_{n=1}^m Ax_n{}^* \otimes x_n \in \mathcal{F}(X, X)$ it is easy to verify that $V^* = AT^*$.
Let $(S_\alpha) \subset \mathcal{F}(X, X)$ be a net such that $S_\alpha \to I_X$ pointwise. Since $X^* \widehat{\otimes}_\pi X$ is a subspace of $X^* \widehat{\otimes}_\pi X^{**}$ for all α, we have $\|VS_\alpha\|_\pi = \|j_X V S_\alpha\|_\pi$. Therefore

$$\begin{aligned}
\|T\|_\pi &= \sum_{n=1}^m (Ax_n{}^*)(x_n) = \lim_\alpha \sum_{n=1}^m (Ax_n{}^*)(S_\alpha x_n) \\
&= \lim_\alpha \sum_{n=1}^m (S_\alpha{}^* A x_n{}^*)(S_\alpha x_n) \\
&= \mathrm{limtrace}(VS_\alpha) \leq \limsup_\alpha \|VS_\alpha\|_\pi = \limsup_\alpha \|j_X V S_\alpha\|_\pi .
\end{aligned}$$

□

Next, we proof the main result under this section.

Theorem 7.4. *Let X be a Banach space, I an integral operator and $1 \leq \lambda < \infty$.If X has λ-Bounded approximation property for I then it has λ-bounded approximation property.*

Proof. Since X has λ-Bounded approximation property for I, then for every $l_1(\Gamma)$-space and for every $T \in \mathrm{I}(\mathrm{X}, l_1(\Gamma)^{**})$ there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ pointwise and limsup $_\alpha \|TS_\alpha\|_1 \leq \lambda \|T\|_{\mathrm{I}}$.Since $TS_\alpha \in \mathcal{F}(X, l_1(\Gamma)^{**}) = X^* \otimes l_1(\Gamma)^{**}$ and $l_1(\Gamma)^{**}$ has the metric approximation preoperty, it is well known that $\|TS_\alpha\|_\pi = \|TS_\alpha\|_{\mathcal{N}} = \|TS_\alpha\|_I$. Recalling that every Banach space is a quotient of some $l_1(\Gamma)-$ space we may assume that for every $U \in \mathrm{I}(X, X^{**})$ there exists (S_α) as above such that $\limsup_\alpha \|US_\alpha\|_\pi \leq \lambda \|U\|_{\mathrm{I}}$. Let $T \in \mathcal{F}(X, X)$. Choose A and V. Then choose a net $(S_a) \subset \mathcal{F}(X, X)$ to be pointwise convergent to I_X such that $\limsup \|j_X V S_\alpha\|_\pi \leq \lambda \|j_X V\|_I$. Then by

$$\|T\|_\pi \leq \lambda \|j_X V\|_1 = \lambda \|V\|_I$$

On the other hand, since $V^* = AT^*$,

$$\|V\|_I = \|V^*\|_I = \|AT^*\|_I \leq \|T^*\|_I = \|T\|_I$$

In conclusion,

$$\|T\|_\pi \leq \lambda \|T\|_I$$

Which means that X has the λ-Bounded approximation property. □

8 The Nuclear Operator and the Weak Bounded Approximation Property

Theorem 8.1. *Let X be a Banach space, and $1 \leq \lambda < \infty$. The following statements are equivalent.*

(a) X has the weak $\lambda - BAP$

(b) $\|T\|_\pi \leq \lambda \|j_X T\|_{\mathcal{N}}$, for all $T \in \mathcal{F}(X, X)$.

We will also need a reformation of the weak bounded approximation property in terms of extension operators as in the Theorem that follows.

Definition 8.1. Let X be closed subspace of a Banach space $W\cdot$ An operator $\Phi \in L(X^*, W^*)$ is called an extension operator if $(\Phi x^*)(x) = x^*(x)$ for all $x \in X$ and $x^* \in X^*$.

Theorem 8.2. *Let X be a Banach space, and $1 \leq \lambda < \infty$. The following statements are equivalent.*

(a) X has the weak $\lambda - BAP$.

(b) There exists an extension operator $\Phi \in \overline{X \otimes X^*}^{w^*} \subset L(X^*, X^{***}) = \left(X^* \widehat{\otimes}_\pi X^{**}\right)^*$ with $\|\Phi\| \leq \lambda$.

Remark 8.1. $AT \in X^* \widehat{\otimes}_\pi X^{**}$ is defined in the usual way if $T = \sum_n {x_n}^* \otimes u_n$, with ${x_n}^* \in X^*, u_n \in l_1$, then $AT = \sum_n {x_n}^* \otimes A u_n$.

Theorem 8.3. *Let X be a Banach space, I an integral operator and $1 \leq \lambda < \infty$. The following statements are equivalent.*

(a) X has the weak $\lambda - BAP$

(b) has the $\lambda - BAP$ for $\mathcal{N}$.

Proof. We first establish (b) for $Y = l_1$ since the nuclear operators factor through l_1 and the dual space of $\mathcal{N}(X, l_1) = X^* \widehat{\otimes}_\pi l_1$. Let Φ be the extension operator as defined and let $(S_v) \subset \mathcal{F}(X, X)$ be a net such that ${S_v}^* \to \Phi$ weak * in $L(X^*, X^{***}) = \left(X^* \widehat{\otimes}_\pi X^{**}\right)^*$. Let $T \in \mathcal{N}(X, l_1) = X^* \widehat{\otimes}_\pi l_1$. We may assume without loss of generality that $\|T\|_\pi = 1$. We show that every compact subset K of X and for every $0 < \varepsilon$ the convex subset $C = \{TS : S \in \mathcal{F}(X, X), \|Sx - x\| \leq \varepsilon$ for all $x \in K\}$ of $X^* \widehat{\otimes}_\pi l_1$ intersects the closed ball $B = \left\{u \in X^* \widehat{\otimes}_\pi l_1 : \|u\|_\pi \leq \lambda + \varepsilon\right\}$.
If these were not the case then, there would exist $A \in L(l_1, X^{**}) = \left(X^* \widehat{\otimes}_\pi l_1\right)^*$ with $\|A\| = 1$ such that

$$\begin{aligned}\lambda + \varepsilon = \sup\{\mathrm{Re}\langle A, u\rangle : u \in B\} &\leq \inf\{\mathrm{Re}\langle A, TS\rangle : TS \in C\} \\ &\leq \lim_v |\langle A, TS_v\rangle| = \lim_v |\langle {S_v}^*, AT\rangle| = |\langle \Phi, AT\rangle| \leq \lambda \|AT\|_\pi \leq \lambda\end{aligned}$$

a contradiction which establishes $Y = l_1$.
Next we let Y be a Banach space, let $T \in \mathcal{N}(X, Y)$ and let $\varepsilon > 0$. According to [5] there exists $R \in (l_1, Y)$ and $\hat{T} \in \mathcal{N}(X, l_1)$ with $\|R\| \leq 1$ and $\|\hat{T}\|_{\mathcal{N}} \leq \|T\|_{\mathcal{N}} + \varepsilon/\lambda$ Such that $T = R\hat{T}$. Let $(S_\alpha) \subset \mathcal{F}(X, X)$ be a net such that $S_\alpha \to I_X$ uniformly on compact sets and Next, we show (b) $\to$ (a) Let $T \in \mathcal{F}(X, X)$. Choose A and V. Then $V^* = AT^*$ and For every net $(S_\alpha) \subset \mathcal{F}(X, X)$ converging pointwise to I_X. Since $j_X V \in \mathcal{N}(X, X^{**})$, for every $\varepsilon > 0$, we can write $j_X V = \sum_{n=1}^\infty {x_n}^* \otimes {x_n}^{**}, {x_n}^* \in X^*, {x_n}^{**} \in X^{**}$, With $\sum_{n=1}^\infty \|{x_n}^*\| \|{x_n}^{**}\| < \|j_X V\|_{\mathcal{N}} + \varepsilon$ Now choose an $l_1(\Gamma)$-space such that X is its quotient space and denote $q : l_1(\Gamma) \to X$ the quotient mapping. q^* Will be an isometric embedding hence for all ${x_n}^{**}$, there exists ${u_n}^{**} \in l_1(\Gamma)^{**}$ such that $q^{**}{u_n}^{**} = {x_n}^{**}$ and $\|{u_n}^{**}\| = \|{x_n}^{**}\|$. Define And choose a net $(S_\alpha) \subset \mathcal{F}(X, X)$ converging pointwise to I_X such that $\lim_\alpha \sup \|US_\alpha\|_{\mathcal{N}} \leq \lambda \|U\|_{\mathcal{N}} \leq \lambda \sum_{n=1}^\infty \|{x_n}^*\| \|{u_n}^{**}\|$ Moreover,

$$\|j_X V\|_{\mathcal{N}} = \|V^*\|_{\mathcal{N}} = \|AT^*\|_{\mathcal{N}} \leq \|T^*\|_{\mathcal{N}} = \|j_X T\|_{\mathcal{N}}.$$

On the other hand, it can be easily verified that $j_X V = q^{**}U$.Hence $j_X V S_\alpha = q^{**} U S_\alpha$. Therefore,

$$\|j_X V S_\alpha\|_\pi = \|q^{**} U S_\alpha\|_\pi \leq \|U S_\alpha\|_\pi = \|U S_\alpha\|_{\mathcal{N}}.$$

In conclusion

$$\|T\|_\pi \leq \lim_\alpha \sup \|j_X V S_\alpha\|_\pi \leq \lim_\alpha \sup \|U S_\alpha\|_{\mathcal{N}} < \lambda \|j_X T\|_{\mathcal{N}} + \varepsilon$$

By letting $\varepsilon \to 0$, we have $\|T\|_\pi < \lambda \|j_X T\|_{\mathcal{N}}$ which means that X has weak Bounded approximation property as required. □

9 Conclusion

The study determines some spaces of ideal operators which hold both the space and ideal characteristics by the use of approximation properties. The u-ideals and their local and hereditary properties have been studied . Using the Hahn-Banach extension, a study on strict u-ideals is done and shown that if there is a Banach space which is a strict u-ideal and has a separable subspace, then the separable subspace is a strict u-ideal and it is separably determined. The results give the metric approximation property in classes of tonsorially well defined Hahn-Banach extensions and equivalent renorming. We have given the renorming of $u-$ideals as an operator space and presented the integral property, $\lambda-$boundedness and $\lambda-$boundedness approximation property and established the connectedness of the three properties.Indeed there is established the equivalence of nuclearity and weak $\lambda-$boundedness.

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