On Some Aspects of Compactness in Metric Spaces

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ABSTRACT

In this paper, we investigate the generalizations of the concepts from Heine-Borel Theorem and the Bolzano-Weierstrass Theorem to metric spaces. We show that the metric space X is compact if every open covering has a finite subcovering. This abstracts the Heine-Borel property. Indeed, the Heine-Borel Theorem states that closed bounded subsets of the real line $\mathbb{R}$ are compact. In this study, we rephrase compactness in terms of closed bounded subsets of the real line $\mathbb{R}$, that is, the Bolzano-Weierstrass theorem. Let X be any closed bounded subset of the real line. Then any sequence (x_n) of the points of X has a subsequence converging to a point of X. We have used these interesting theorems to characterize compactness in metric spaces.

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1 Introduction

Compactness is a property in metric spaces. A metric space (X, ρ) is said to be totally bounded (or precompact) if, for every $\epsilon > 0$, the space X can be covered by a finite family of open balls of radius ϵ [5]. A metric space X is said to be sequentially compact if every sequence $(x_n)_{n=1}^{\infty}$ of points in X has a convergent subsequence [6]. This abstracts the Bolzano-Weierstrass property, that is, closed bounded subsets of the real line are sequentially compact.

If X is a non-void set and E a subset of X; a family $\{E_\alpha : \alpha \in \Lambda\}$ is said to be a cover for or of E [4]. If $\bigcup_{\alpha\in\Lambda} E_\alpha \supseteq E$. If τ is a topology in X and each E_α is open in (X, τ), then we call $\{E_\alpha : \alpha \in \Lambda\}$ an open cover for E; If Λ' is a non-void subset of Λ and $\{E_\alpha : \alpha \in \Lambda'\}$ also covers E, then $\{E_\alpha\} : \alpha \in \Lambda'$ is called subcover for E.

Definition 1. Let (X, τ) be a topological space, we say that (X, τ) is compact if every open cover of X contains a finite subcover. If Y is a non-void subset of X, then Y is compact if the subspace (Y, τ_y) (where τ_y is the induced topology in Y) is compact.

Most definitions in this paper can be found in [1], [3] and [2].

2 Compactness in Metric Spaces

We have the following fundamental results characterizing compact metric spaces:

Proposition 1. *A metric space is sequentially compact if and only if it has the finite intersection property for closed sets.*

Proof. Suppose that X is sequentially compact. Given a decreasing sequence of closed sets F_n, choose $x_n \in F_n$ for each $n \in \mathbb{N}$. Then (x_n) has a convergent subsequence (x_{n_k}) with $x_{n_k} \to x$ as $k \to \infty$. Since $x_{n_k} \in F_n$ for all $n_k \geq n$ and F_n is closed, $x \in F_n$ for every $n \in \mathbb{N}$, so $x \in \bigcap_{n=1}^{\infty} F_n$, and $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Conversely, suppose that X has the finite intersection property. Let (x_n) be a sequence in X and define

$$F_n = \overline{T_n}, \quad T_n = \{x_k : k > n\}.$$

Then (F_n) is a decreasing sequence of non-empty, closed sets, so there exists

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

Choose a subsequence (x_{n_k}) of (x_n) as follows. For $k = 1$, there exists $x_{n_1} \in T_1$ such that $d(x_{n_1}, x) < 1$, since $x \in F_1$ and T_1 is dense in F_1. Similarly, since $x \in F_{n_1}$ and T_{n_1} is dense in F_{n_1}, there exists $x_{n_2} \in T_{n_1}$ with $n_2 > n_1$ such that $d(x_{n_2}, x) < 1/2$. Continuing in this way (or by induction), given x_{n_k} we choose $x_{n_{k+1}} \in T_{n_k}$, where $n_{k+1} > n_k$, such that $d\left(x_{n_{k+1}}, x\right) < 1/(k+1)$. Then $x_{n_k} \to x$ as $k \to \infty$, so X is sequentially compact. □

Proposition 2. *A metric space is compact if and only if it is sequentially compact.*

Proof. Suppose that X is compact. Let (F_n) be a decreasing sequence of closed nonempty subsets of X, and let $G_n = F_n^c$.

If $\bigcup_{n=1}^{\infty} G_n = X$, then $\{G_n : n \in \mathbb{N}\}$ is an open cover of X, so it has a finite subcover $\{G_{n_k} : k = 1, 2, \ldots K\}$ since X is compact. Let

$$N = \max\{n_k : k = 1, 2, \ldots K\}.$$

Then $\bigcup_{n=1}^{N} G_n = X$, so

$$F_N = \bigcap_{n=1}^{N} F_n = \left(\bigcup_{n=1}^{N} G_n\right)^c = \emptyset,$$

contrary to our assumption that every F_n is nonempty. It follows that $\bigcup_{n=1}^{\infty} G_n \neq X$ and then

$$\bigcap_{n=1}^{\infty} F_n = \left(\bigcup_{n=1}^{\infty} G_n\right)^c \neq \emptyset,$$

meaning that X has the finite intersection property for closed sets, so X is sequentially compact. Conversely, suppose that X is sequentially compact. Let

$$\{G_\alpha \subset X : \alpha \in I\}$$

□

Proposition 3. *Let (X, τ) be a topological space and Y be a non-void subset of X. Then Y is compact if and only if for any open cover $\{G_\alpha : \alpha \in \Lambda\}$ (of subsets of X) for Y there exists a finite number of indices $\alpha_1, \ldots \alpha_n$ in Λ such that*

$$Y \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

Proof. (i) Suppose Y is compact, i.e. (Y, τ_y) is compact. Let $\{G_\alpha : \alpha \in \Lambda\}$ be any family of open subsets of X which cover Y i.e.

$$Y \subseteq \bigcup_{\alpha \in \Lambda} G_{\alpha_i}$$

For each $\alpha \in \Lambda, G_x \cap Y$ is open in (Y, τ_Y) and clearly

$$Y \subseteq \left(\bigcup_{\alpha \in \Lambda} G_\alpha\right) \cap Y = \bigcup_{\alpha \in \alpha} (G_\alpha \cap Y)$$

Thus $\{G_\alpha \cap Y : \alpha \in \Lambda\}$ is a family of open subsets of (Y, τ_Y) which covers Y.

But Y is compact. Hence there exist a finite number of indices $\alpha_1 \ldots \alpha_n \in \Lambda$ such that

$$Y \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y) = \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap Y$$

$$\text{i.e.} \quad Y \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

Conversely, let $\{G_\alpha : \alpha \in \Lambda\}$ be any collection of open subsets of (X, τ) such that

$$\bigcup_{\alpha \in \Lambda} G_\alpha \supseteq Y \tag{2.1}$$

By hypothesis, there exists a finite number of indices $\alpha_1, \ldots \alpha_n \in \Lambda$ such that

$$\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq Y \tag{2.2}$$

Since G_α is open in (X, τ). So $G_\alpha \cap Y$ is open in (Y, τ_Y) for each $\alpha \in \Lambda$.
From (1.2) we have

$$Y\left(\bigcup_{\alpha\in\Lambda} G_\alpha\right) \supseteq Y \text{ i.e. } \bigcup (G_\alpha \cap y) \subseteq Y$$

So $\{G_\alpha \cap Y : \alpha \in \Lambda\}$ is an open cover for Y and this family consists of subsets of Y. So $\bigcup_{\alpha\in\Lambda}(G_x \cap Y) \subseteq Y$.
Thus $\bigcup_{\alpha\in\Lambda}(G_\alpha \cap Y) = Y$ and likewise (1.3) gives

$$\bigcup_{i=1}^{n}(G_{\alpha_i} \cap Y) = Y$$

Thus (Y, τ_Y) is compact for it is clear that every open cover of T by subsets of Y has a finite subcover.
The second part can be seen more clearly thus:
Let $\{E_\alpha : \alpha \in \Lambda\}$ be a family of open subsets of Y which covers Y i.e.

$$\bigcup_{\alpha\in\Lambda} E_\alpha \supseteq Y$$

Since E_α is open in (Y, τ_Y) there exist a subset $G_\alpha \subseteq X$ such that G_α is open in (X, τ) and $G_\alpha \cap Y = E_\alpha$.
Therefore,

$$\bigcup_{\alpha\in\Lambda}(G_\alpha \cap Y) \supseteq Y \text{ i.e.} (\bigcup_{\alpha\in\Lambda} G_\alpha) \cap Y \supseteq Y$$

By hypothesis, there exist a finite number of indices $\alpha_1, \ldots, \alpha_n$ in Λ such that

$$\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq Y$$

Therefore, $\bigcup_{i=1}^{n}(G_{\alpha_i} \cap Y) = Y$ (as seen above) i.e. $\bigcup_{i=1}^{n} E_{\alpha_i} = Y$ i.e. (Y, τ_Y) is compact. □

Proposition 4. *Let (X, ρ) be a metric space and be compact. Then (X, ρ) is bounded. Thus every compact metric space is bounded.*

Proof. Let $\epsilon > 0$ be arbitrary given. Then $\{N(x; \varepsilon) : x \in X\}$ is an open cover for X; Since

$$X = \mathrm{U}_{x\in\alpha} N(x; \varepsilon) \text{ for } x \in N(x; \varepsilon) \text{ so } \{X\} \subseteq N(x; \varepsilon) \subseteq X$$

Therefore, $\bigcup_{i=1}^{n}\{X\} \subseteq \bigcup_{i=1}^{n} N(x; \varepsilon) \subseteq X$. Since (X, ρ) is compact, there exist a finite number of points $x_1, \ldots, x_n \in X$ such that $\{N(x_i, \varepsilon) : 1 \leqslant i \leqslant n\}$ is a subcover for X, i.e.

$$X = \bigcup_{i=1}^{n} N(x_i, \epsilon)$$

Let $A = \{x_1, \cdots, x_n\}$. Then A being a finite subset of X, is bounded. In fact,

$$\text{diam A} = \sup\{\rho(x, y) : x, y \in A\}$$

$$= sup\, \{\rho\,(\lambda_i; x_j) : 1 \leq i, j \geq n\}$$

Let $(x, y) \in X$. Then $x \in N\,(x_i; \varepsilon)$ an $y \in N\,(x_i; \varepsilon)$ for some i, j where $1 \leq i, j \leq n$. Now

$$\text{therefore, } \rho(x, y) \leq \rho(x, x_i) + \rho(x_i, x_j) + \rho(x_j, y)$$

$$< \varepsilon + \rho\,(x_i, x_j) + \varepsilon.$$

$$\leq 2\varepsilon + diam(A)$$

therefore$sup\{\rho(x, y) : x, y \in X\} \leq \underbrace{2\varepsilon + \mathrm{diam}(A) < +\infty}_{A \text{ is bounded}}$

i.e. diam $(x) < +\infty$ i.e. X is bounded. □

Corollary 1. *If E is a compact subset of a metric space (X, ρ) then E is bounded.*

Proof. E is compact implies (E, ρ_E) is compact which implies E is bounded. □

However, bounded subsets of a metric space need not be compact.

Counter Example 1. *Let X be a set of infinite cardinality and ρ be the discrete metric in X, i.e.*

$$\rho(x, y) = 1 \text{ if } x, y \in X \text{ and } x \neq y$$
$$0 \text{ if } x = y$$

therefore $\sup\{\rho(x, y) : xy \in X\} = \mathrm{diam}\, X = 1$ i.e. X is bounded.
Every singleton set $\{x\}$ (where $x \in X$) is open in (X, ρ). Indeed, let $0 < \varepsilon \leq 1$. Now $X \in \{x\}$ and

$$N(x; \varepsilon) = \{x\} \text{ for } y \in X \text{ and } y \neq x \Rightarrow \rho(y, x) = 1$$
$$\text{so } y \notin N(x; \varepsilon)$$

Thus $N(x; \varepsilon) \subseteq \{x\}$ i.e. x is an interior point of $\{x\}$. Thus $\{x\}$ is open in (X, ρ) for each point of $\{x\}$ an interior point since $\bigcup_{x \in X}\{x\} = X$ so $\{\{x\} : x \in X\}$ is an open cover for X.
Suppose X is compact. Then there exist a finite number of points $x_1, \ldots, x_n$ such that $\bigcup_{i=1}^{n}\{x_i\} = X$ i.e. $X = \{x_1, \ldots, x_n\}$ a contradiction !! since X is of infinite cardinality. Hence the supposition that X is compact is not valid. Hence X is not compact.

Proposition 5. *Compact subsets of a metric space are closed.*

Proof. Let E be a compact subset of a metric space (X, ρ). We shall show that $E^c (= X - E)$ is open in (X, ρ). Let $y \in E^C$. So $y \notin E$. Let $x \in E$. Then $y \neq x$, so $\rho(y, x) = \varepsilon_x > 0$. Then

$$N\left(y; \frac{\varepsilon_x}{2}\right) \cap N\left(x; \frac{\varepsilon_x}{2}\right) = \phi$$

As we vary x over E (keeping y fixed) we get a system of neighborhoods $\left\{N\left(y; \frac{\varepsilon_x}{2}\right) : x \in E\right\}$ all centred at y and a system of neighborhoods.

$$\left\{N\left(x; \frac{\varepsilon_x}{2}\right) : x \in E\right\}. \text{ Now}$$

$$E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} N\left(x; \frac{\varepsilon_x}{2}\right).$$

Thus the family $\left\{N\left(x; \frac{\varepsilon_x}{2}\right) : x \in E\right\}$ is an open cover for E.
Since E is compact, there exists a finite number of points say $x_1, \ldots, x_n \in E$ such that

$$E \subseteq \bigcup_{i=1}^{n} N\left(x_i; \frac{\varepsilon_{x_i}}{2}\right)$$

Consider the subfamily $\left\{N\left(y; \frac{\varepsilon_x}{2}\right) : 1 \leqslant i \leq n\right\}$ of $\{N\left(y; \varepsilon_x\right) : x \in E\}$
Let $\varepsilon = \frac{1}{2} min \left\{\varepsilon_{x_1}, \varepsilon_{x_2}, \cdots \ldots, \varepsilon_{x_n}\right\}$ since $n \in \mathbb{N}$ and each $\varepsilon_x > 0,$ so $\varepsilon > 0$.Now

$$N(y; \varepsilon) = \bigcap_{i=1}^{n} N\left(y; \frac{\varepsilon_{x_i}}{2}\right) \tag{2.3}$$

Since $N\left(y; \frac{\varepsilon_x i}{2}\right) \cap N\left(x_i; \frac{\varepsilon_x i}{2}\right) = \phi$ for each $i = 1, 2, \ldots n$.
It follows from (1.4) that

$$N(y; \varepsilon) \cap N\left(x_i; \frac{\varepsilon x_i}{2}\right) = \phi$$

for each $\quad i = 1, \ldots n$ i.e.

$$N(y : \varepsilon) \cap \left(\bigcup_{i=1} N\left(x_i; \frac{\varepsilon_{x_i}}{2}\right) = \phi\right)$$

$$\text{But } E \subseteq \bigcup_{i=1}^{n} N\left(x_i; \frac{\varepsilon_{x_i}}{2}\right)$$

Therefore, $N(y; \varepsilon) \cap E = \phi$ i.e. $N(y; \varepsilon) \subseteq E^c$. Thus $y \in E^c$ and $N(y; \varepsilon) \subseteq E^c$, so y is an interior point of E^c. Therefore, E^c is open in (X, ρ) i.e. E is closed in (X, ρ). □

Corollary 2. *Let E be a closed subset of a metric space X and F be a compact subset. Then $E \cap F$ is compact subset of X.*

Proposition 6. *Let E be a compact subset of a metric space X and F be a closed subset of E. Then F is compact.*

Proof. Let $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover for F. Since F is closed in X, F^c is open in X. Adjoin F^c to the collection $\{G_\alpha : \alpha \in \Lambda\}$ and we obtain the family $\{G_\alpha : \alpha \in \Lambda\} \cup \{F^c\}$ which clearly covers E. But E is compact. Hence there exist a finite number of members of the family.

$$\{G_\alpha : \alpha \in \Lambda\} \cup \{F^c\}.$$

which is an open subcover for E and hence also an open cover for F (since $F \subset E$). Delete F^{C} from this finite subcover for E if F^{c} belongs to it and we are thus left with a finite collection of members of $\{G_{\alpha} : \alpha \in, \Lambda\}$ which cover F. Thus F is compact. □

Corollary 3. *Let X be a metric space, E a closed subset of X and F a compact subset of X. Then $E \cap F$ is compact.*

Proof. F is compact implies F is closed. E is closed, F is closed implies $E \bigcap F$ is closed.
Now $E \cap F \subset C$, a compact set and $E \cap F$ is closed. Thus $E \cap F$ is a closed subset of a compact set F. Therefore, $E \cap F$ is compact.

□

Definition 2. Let (X, ρ) be a metric space and E be a subset of X. Let $\varepsilon > 0$ be a real number. A subset A of X is called an $\varepsilon - net$ for E if

$$E \subseteq \bigcup_{y \in A} N(y; \varepsilon)$$

The subset E is said to be precompact or totally bounded if for every real $\varepsilon > 0, E$ has a finite ε-net.

Proposition 7. *Let (X, ρ) be a metric space and E be a precompact subset of X. Then E is bounded.*

Proof. E is precompact implies for each real $\varepsilon > 0$, E has a finite $\varepsilon - net$.
E is precompact implies that for each real $\varepsilon > 0$, E has a finite ε-net.
$A_{\varepsilon} = \{x_1, \cdots, x_n\}$ say, where $x_i \in X(i = 1, \ldots n)$
Therefore, $E \subseteq \bigcup_{i=1}^{n} N(x_i; \varepsilon)$ Let $x, y \in E$. Then there exist $i, j (1 \leq i, j \leq n)$ such that $x \in N(x_i; \varepsilon), y \in N(x_j; \varepsilon)$ so $\rho(x, y) \leq \rho(x, x_i) + \rho(x_i, x_j) + \rho(x_j, y) < 2\varepsilon + \rho(x_i, x_j) \leq 2\varepsilon + \operatorname{diam} A_{\varepsilon} < +\infty$. (for A_{ε} is a finite subset of X so $\operatorname{dim} A_{\varepsilon} < +\infty$) Hence sup $\{\rho(x, y) : x, y \in E\} \leqslant 2\varepsilon + \operatorname{diam} A_{\varepsilon} < +\infty$
i.e. E is bounded.
However, boundedness does not imply precompactness. □

Counter Example 2. *Consider the Hilbert space ℓ^2 and let*

$$A = \left\{x \in \ell^2 : \|x\| = 1\right\}$$

Clearly A is bounded for if $x, y \in A$, then

$$||x - y\| \leq \|x\| + \|y\| = 1 + 1 = 2$$

consider the elements $(e_n)_{n=1}^{\infty}$ defined by

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots) \\ e_2 &= (0, 1, 0, \cdots) \\ &\vdots \\ e_n &= (0, 0, \ldots .0, 1, \cdots) \end{aligned}$$

Since $\|e_n\| = 1 \forall n \in \mathbb{N}$*, it is clear that* $e_n \in A \; \forall \; n \in \mathbb{N}$.

Also if $m \neq n$

$$\|e_n - e_n\|^2 = \|e_n\|^2 + \|e_m\|^2 - \langle e_n e_n\rangle - \langle e_{m1} e_n\rangle$$

Since $m \neq n \Rightarrow \langle em, e_n\rangle = 0$
$\therefore \|e_n - e_n\| = \sqrt{2}$

Take an ε *satisfying* $0 < \varepsilon < \sqrt{2}/2$.
Let y_i, y_j *be elements such that* $\|e_n - y_j\| < \varepsilon$ *and* $\|e_m - y_j\| < \varepsilon$.
Then $y_i \neq y_j$ *Since*

$$y_i = y_i \Rightarrow \rho(e_m, e_n) \leq \|e_m - e_n\| \leq \rho(e_m, y_j) + \rho(y_j, y_i) + \rho(y_i, e_n) < \varepsilon + \varepsilon = 2\varepsilon < \sqrt{2} \tag{2.4}$$

which is impossible. Since $m \neq n$ $(\|e_n - \ln\| = \sqrt{2}$ *if* $m \neq n)$.
Suppose A *is precompact. Then* A *must have a finite* $\varepsilon -$ *net. Since this finite* $\varepsilon -$ *net cover* A *and* $e_n \in A \; \forall \; n \in N$*. So an infinite number of* e_n *must belong to some* $N(y; \varepsilon)$ *for a y belonging to their net. But this would imply (1.5), which is impossible. Hence the supposition that* A *is precompact is not correct. Therefore* A *is not precompact.*
Thus boundedness implies precompactness.

Proposition 8. A *is compact implies* A *is precompact.*

Proof. Let $\varepsilon > 0$ be any given real number. Then (if x is entire metric space) the family $\{N(x; \varepsilon) : x \in X\}$ covers A. Since A is compact, there exist a finite number of points $x_1, \ldots x_n(say)$ in X such that

$$A \subseteq \bigcup_{l=1}^{n} N(x_i; \varepsilon)$$

Thus $\{x_1, \ldots x_n\}$ is a finite $\varepsilon -$ net for A. This shows that A is precompact. □

Proposition 9. *Let* (X, ρ) *be a metric space and* A *be a non-void subset of* X*.*
If every sequence of points of A has a convergent subsequence; then A *is precompact.*

Proof. We have to prove
(every sequence of points of A has a convergent subsequence implies A is precompact).
For this it is sufficient to establish A is not precompact implies there is a sequence (x_n) of points of A which has no convergent subsequence. Suppose A is not precompact. Hence there is a real $\varepsilon > 0$ for which there is no finite $\varepsilon - net$ for A. Choose $x_1 \in A$.

Since $\{x_1\}$ is not a $\varepsilon -$ net for A there exits a point $x_2 \in A$ such that $x_2 \notin N(x, \varepsilon)$ i.e. $\rho(x_1, x_2) \geq \varepsilon$.

Now $\{x_1, x_2\}$ is not an ε-net for A; hence there exist $x_3 \in A$ such that $x_3 \neq x_1, x_3 \neq x_2$ and hence $x_3 \notin N(x_1; \varepsilon) \cup N(x_2; \varepsilon)$ hence $\rho(x_1, x_3) \geq \varepsilon, \rho(x_2, x_3) \geq \varepsilon$ (Note also $\rho(x_1, x_2) \geq \varepsilon$). Continuing in this manner we would obtain an infinite sequence (x_n) of distinct points of A such that $\rho(x_i, x_j) \geqslant \varepsilon$ whenever $i \neq j$.
(Note that no finite sub-collection of $\{x_n : n \in N\}$ can be an $\varepsilon -$ net for A since A is not precompact).
Since $\rho(x_i, x_j) \geqslant \varepsilon \quad \forall i \neq i$, there does not exist any convergent subsequence of (x_n). □

Remark 1. *From the above proof it not only follows that if every sequence of points of A has a convergent subsequence then A is precompact but also that the finite E-net for A (for each real $\varepsilon > 0)$ consist of points of A.*

3 Conclusion

Since the concept of compactness plays a central role in functional analysis and indeed in all areas of analysis it is important for us to obtain some intuition about when sets are or are not compact. The results in this paper provide a basis for carrying out analysis in metric spaces. We have shown that a metric space X is compact, that is, every open covering of X has a finite subcovering. Also, every collection of closed sets in X with the finite intersection property has a nonempty intersection. If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ is a decreasing sequence of nonempty closed sets in X, then $\bigcap_{n=1}^{\infty} F_n$ is nonempty. X is sequentially compact, that is, every sequence in X has a convergent subsequence. finally, X is totally bounded and complete.

AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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