ON REGULAR ELEMENTS AND VON-NEUMANN INVERSES OF ZERO-SYMMETRIC LOCAL NEAR-RINGS WITH JORDAN IDEALS ADMITTING FROBENIUS DERIVATIONS

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TITLE PAGE

DECLARATION

This research thesis is my original work prepared with no other than the indicated sources and support and has not been presented elsewhere for a degree or any other award.

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CERTIFICATION

We the undersigned certify that we have read and hereby recommend for acceptance of Masinde Muliro University of Science and Technology a research thesis entitled, "On Regular Elements and Von-Neumann inverses of Zero-symmetric Local Near-rings with Jordan Ideals admitting Frobenius Derivations"

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DEDICATION

To all my family members, I dedicate this work for their support and encouragement throughout the course. Without their support it may not have been easier to accomplish this.

ACKNOWLEDGEMENT

I begin by thanking the Almighty God for His protection throughout the course of my study. I would also like to take this opportunity to very sincerely thank my supervisors Dr. Ojiema Michael Onyango and Prof. Kivunge M. Benard for everything that I have learned from them during the course of this study. I truly express my gratitude to them for their guidance, advice and encouragement throughout this research. I extend my sincere gratitude to all the members of the department of mathematics of Masinde Muliro University of Science and Technology for the support they also gave me in the course of my study. Grateful acknowledgement is also made to my friends, Dr. Isaac Okwany and Mr. Evans Mogoi for their continued encouragement.

ABSTRACT

The study of near-rings with identity is very vital in generalizing characterization of commutative rings with identity. Much of the recent works on the classification of finite rings with identity have considered a characterization paradigm using the unit groups, the zero divisor graphs, adjacency and incidence matrices among others. This has left the non-linear aspects fairly untouched. In particular, regular elements and Von-Neumann inverses of near rings admitting derivations hardly exist in literature. Thus, the study determined the structures of classes of zero symmetric local near rings $\mathcal{N}$ with $n-$nilpotent radical of Jordan ideals, $J(\mathcal{N})$; $n = 2,\ n \geq 3$ with char $\mathcal{N}$ as p, p^2 and p^k; $k \geq 3$ convoluted with Frobenius derivations, the commutation over $\mathcal{N}$ constructed and finally characterized $\mathcal{N}, R(\mathcal{N}), \Gamma(\mathcal{N})$ and the inverses of $\mathcal{N}$. To achieve these, the research used idealization of R_0-modules with respect to Galois rings and Raghavendran's characterization method to construct the classes of near-rings under investigation, the theorems of Asma and Inzamam to determine the commutation over $\mathcal{N}$ via $J(\mathcal{N})$ and the Frobenius derivations, the fundamental theorem of finitely generated abelian groups to determine the structures of $R(\mathcal{N})$ and their inverses and SONATA. The results of this study showed two constructions of classes of zero symmetric local near rings with a Jordan ideal containing a 2-nilpotent radical which admit a commuting Frobenius derivation, determined some graph morphisms which form symmetric groups, the regular elements obtained have structures isomorphic to cyclic groups. The Von-Neumann inverses of the $\mathcal{N}$ formulated agreed with the number theoretic standards of the Von-Neumann inverses of idealized local rings while the arithmetic function, $V(\mid R(\mathcal{N}) \mid)$ followed the asymptotic properties of $V(n)$, $\tau(n)$, $\overline{\omega}(n)$, $\sigma(n)$ and $K(n)$. Furthermore, the results determined the automorphisms of $R(\mathcal{N}$ both in terms of structures and orders.

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R-Commutative ring with identity.

$J(\mathcal{N})$ - Jordan ideal

d-Derivation.

$\mathcal{N}$ - Near-ring.

$\mathcal{N}^*$ - Invertible elements of $\mathcal{N}$.

$\mathbb{Z}_n$ - The ring of integers modulo n.

$\varphi(n)$ - Euler's phi function.

$C(\mathcal{N})$ - Multiplicative center.

R - A finite ring.

R^* - Invertible elements of R.

$R_0 = G\mathcal{N}(p^{kr}, p^k)$ - Galois near-ring of order p^{kr} and characteristic p^k.

$\mathcal{M}$ - Near-module of R_0.

$\mathcal{J}$ - near-ideal of $\mathcal{M}$.

$\mathcal{T}$ - Finite direct product of classes of near-rings.

$\mathcal{S}$ - Finite direct product of family of classes of near-rings.

$f(A)$ - Map or Morphism of a matrix A.

$char\mathcal{N}$ - Characteristic of $\mathcal{N}$.

$Ann(\mathcal{M})$ - Annihilator of $\mathcal{M}$.

$Z_R(\mathcal{N})$ - Right zero divisor.

$Z_L(\mathcal{N})$ - Left zero divisor.

L - Subgroup of a local near-ring.

$\Gamma(\mathcal{N})$ - Graph of $\mathcal{N}$.

$Aut(\Gamma(\mathcal{N}))$ - Automorphism of graph of $\mathcal{N}$.

E - Edge of a graph.

$R(\mathcal{N})$ - Regular elements of $\mathcal{N}$.

$\mid R(\mathcal{N}) \mid$ - Number of elements of $\mathcal{N}$.

$V(\mid R(\mathcal{N}) \mid)$ - Arithmetic function of the order of $R(\mathcal{N})$.

$Ref(x)$ - Set of all reflexive inverses of $x \in R(\mathcal{N})$.

$I(x)$ - Set of inner inverses of $x \in R(\mathcal{N})$.

$x \parallel y$ - x is a unitary divisor of y.

GL -General linear group.

Δ - Zero automorphism.

CHAPTER ONE

INTRODUCTION

In this chapter, we give a brief mathematical background, definitions and the concepts that have been useful in our research.

1.1 Mathematical background

Roos [64] was the first to define the concept of regularity for rings. Later certain regularities in associative rings were obtained by other authors[36]. Most of these regularities were defined for near-rings. Authors, such as Groenewald and Potgieter [36] improved large part of the general theory about those regularities. Mason [40] examined the concepts of regular and strongly regular for right near-rings. Furthermore, Mason argued that it is necessary to distinguish between strong left and right regularity. In recent years, Mason proved that for a zero-symmetric near-ring with identity, the notions of left regularity, right regularity and left strong regularity are equivalent. Reddy and Murty [63] have proven that these three notions are equivalent for arbitrary near-rings. Also, Hongan [40] has proven that these three notions and right strongly regular are equivalent. These results among others demonstrate that regularity properties for rings and dual conditions in near-rings have been studied in detail. Indeed, several authors have also researched on the relationships between the concepts of primality and strongly regular. For example, Argaç and Groenewald [6] used left 0-prime and left prime ideals to characterize strongly regular near-rings. Moreover, it was attempted to adapt the concept of strongly regular to the notions of ring and near-ring. Handelman and Lawrence [37] introduced strongly prime rings. Groenewald [35] proposed the idea of strongly

prime near-rings. Booth, et al [18], defined a strongly equiprime near-ring as an alternative definition of a strongly prime near-ring. Some concepts, such as center, idempotent element, identity, right and left permutable, medial, commutative, abelian, internal multiplier in near-rings, have been studied. In [17, 48, 68], the authors developed the basic properties of medial, left permutable, right permutable and commutative near-rings. Furthermore, Mason [40] and Drazin [32] studied the concepts of center and idempotent element and also examined some relationships between these concepts. Also, several authors studied relationships with regular forms, strongly regular forms and prime ideals of these concepts. Birkenmeier [16] examined relationships between sets of idempotent elements and completely semiprime ideals. Mason [50] introduced strong forms of regularity for near-rings and examined some relations between the concepts of idempotent element and strongly regular. Dheena [30] presented a generalization of strongly regular near-rings. Drazin [32] studied regularity in near-rings where all idempotent elements are central. Andrunakievich [4] defined p-regular rings and Choi [24] extended the p-regularity of rings to the p-regularity of near-rings. In 2012, Dheena and Jenila [31] introduced the notion of p-strongly regular near-rings and obtained equivalent conditions for near-rings to be p-strongly regular. They also were the first to define the concept of p-prime [31].
Kamal and Khalid [44] in their study of commutativity of near rings with derivations found that any near-ring admits a derivation iff it is zero-symmetric. They also proved some commutativity theorems for a non-necessarily 3-prime near-ring with a suitably constrained derivation d with the condition that $d(a)$ is not a left zero divisor in R for some $a \in R$. As a consequence, they attempted to advance further research around a classification of 3-prime near-rings admitting derivations.

Near-rings which indeed are generalized rings, need not be commutative, and most importantly, only one distributive law is postulated (e.g., Example 1.4, Pilz

[59]). The pioneer work on derivations on near rings was conducted by Bell and Mason [13] where they characterized the derivations on near rings and near fields from a generalized point of view. Their work was however motivated by the study by Posner [60] concerning derivations on prime rings, mappings that did not have suitable constraints and thus not extrapolated to near-rings. A characterization of the commutativity property of prime and semi-prime near-rings having certain constraints on derivation has been advanced by a number of algebraists, see for example ([7],[9],[12],[26],[33],[38],[60],[61]). A number of dual results have also been obtained for near-rings (cf.[5],[10],[13],[29],[40],[66],[67],[71]). Daif and Bell [26] established the following result: Let $\mathcal{I}$ be a nonzero ideal of a prime ring $\mathcal{R}$. If d is a derivation on $\mathcal{R}$ satisfying $d([v,w]) = \pm[v,w]$ for all $v, w \in \mathcal{I}$, then $\mathcal{R}$ is commutative. Boua and Oukhtite [21] proved that if a 3-prime near ring $\mathcal{N}$ with a nonzero derivation d satisfying either $d([v,w]) = \pm[v,w]$ or $d(v \circ w) = \pm(v \circ w)$ for every $v, w \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring. Further, Boua [19] proved that if $\mathcal{U}$ is a semigroup ideal of a 3-prime near ring $\mathcal{N}$ and d is a derivation on $\mathcal{N}$ satisfying any one of the following conditions: (i) $d([v,w]) = [v,w]$, (ii) $d([v,w]) = [d(v),w]$, (iii) $[d(v),w] = [v,w]$, (iv) $d(v \circ w) = d(v) \circ w$, or (v) $d(v) \circ w = (v \circ w)$ for all $v, w \in \mathcal{U}$, then $\mathcal{N}$ is a commutative ring.

The concept of multiplicative derivation in rings was introduced by Daif [25] and it was inspired by Martindale [49]. Goldmann and Semrl [34], studied these mappings and provided the full description of such mappings (for more details, we refer to [25] and [34]). Thus, a mapping (not necessarily additive) $d : \mathcal{N} \longrightarrow \mathcal{N}$ is known as a multiplicative derivation on a near ring $\mathcal{N}$ if $d(vw) = d(v)w + vd(w)$ for every $v, w \in N$. Let $\mathcal{N}[x] = \mathbb{Z}_2[x]$ be a ring of all real valued continuous functions in indeterminate x over $\mathbb{Z}_2$. Thus a map $d : \mathcal{N}[x] \longrightarrow \mathcal{N}[x]$ was defined by [34] as;

$$d(f)(g) = \begin{cases} f(g)\ln \mid f(g) \mid, & f(g) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is verifiable that $d(gh) = d(g)h + gd(h)$ for all $g, h \in \mathbb{Z}_2$ but $d(g + h) \neq d(g) + d(h)$. It therefore follows that multiplicative derivations need not be additive in a general setting. The most recent works of Asma and Inzamam [8] on commutativity of a 3-prime near ring satisfying certain differential identities on Jordan ideals has given an investigation of near-rings admitting derivation a new twist. In fact, they proposed a number of necessary results that qualify 3-prime near rings with Jordan ideals admitting derivations and multiplicative derivations to be commutative. In particular, they proved the commutativity condition for a 3-prime near ring $\mathcal{N}$ under any one of the following conditions:
(i) $[d_1(u), d_2(k)] = [u, k]$, (ii) $d([k, u]) = [d(k), u]$, (iii) $[d(u), k] = [u, k]$, (iv) $d([k, u]) = d(k) \circ u$, (v) $[d(k), d(u)] = 0$ for all $u \in \mathcal{N}$ and $k \in J$, a Jordan ideal of $\mathcal{N}$, where d, d_1, d_2 are derivations on $\mathcal{N}$. Bell and Daif, [12] showed the following result: If R is a 2-torsion free prime ring admitting a strong commutativity preserving derivation d, that is, d satisfies $[d(v), d(w)] = [v, w]$ for every $v, w \in R$, then R is commutative. This result has been extended by Asma and Inzamam [8] for a 3-prime near ring in two directions. First of all, they considered two derivations instead of one derivation, and secondly, they proved the commutativity of a 3-prime near ring $\mathcal{N}$ in place of a ring R in case of a Jordan ideal of $\mathcal{N}$. These results provided very good basic necessary conditions for studying near-rings. Perhaps based on the recommendations of [8], it would be interesting to investigate whether their results would still hold when another type of derivation is used. On the other hand, little is documented about any class of idealized local near-rings. Osba, Henriksen and Osama [56] conducted a classification survey on combining

local and Von Neumann Regular Rings as a basis upon which the regularity properties of rings and their ideals could be explored. The rings studied in [56] were finite and their Von Neumann inverses gave some asymptotic patterns. Their findings demonstrated how to combine the Von Neumann inverses of classes of rings such as the power series rings and the ring of integers. They however did not count the number of regular elements in a given finite ring nor did they give the structural formulae for the regular elements and the Von Neumann inverses of the specified classes of rings. In a closely related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [55] where they characterized the regular elements in the ring of integers modulo n, $\mathbb{Z}_n$. Furthermore, they studied the arithmetic functions denoted as $V(n)$ and determined the relationship between $V(n)$ and the Euler's phi function, $\varphi(n)$. This gave an extension of the ring theoretic algebra employed in counting the regular elements of $\mathbb{Z}_n$ to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in $\mathbb{Z}_n$, then $a^{(-1)} \equiv a^{\varphi(n)-1}$ $(mod\ n)$. They proposed a criterion for getting the possible Von Neumann inverses in the set of regular elements of $\mathbb{Z}_n$ and explored the asymptotic properties of $V(n)$. Their findings did not consider extensions and idealization using maximal submodules of $\mathbb{Z}_n$ $\forall\ n \in \mathbb{Z}^+$. In order to advance the concept of classification of algebraic structures, the thesis considers generalized rings, the near rings.

1.2 Basic concepts

The definitions below are commonly used in the thesis.

Definition 1.2.1. *([45], Chapter 5, section 5.1) A semigroup is a set with an associative binary operation.*

Definition 1.2.2. *An algebraic structure $\mathcal{N}$ endowed with two binary operations $+$*

and $\cdot$ *such that (i)* $(\mathcal{N},+)$ *is a group, (ii)* $(\mathcal{N},\cdot)$ *is a semigroup and (iii)* $u\cdot(v+w) = u\cdot v + u\cdot w$ *for every* $u,v,w \in \mathcal{N}$ *is called a* **left near-ring**. *Dually, if instead of (iii),* $\mathcal{N}$ *satisfies the right distributive law, then* $\mathcal{N}$ *is said to be a* **right near ring**.

Definition 1.2.3. *A near-ring* $\mathcal{N}$ *is called* **local** *if the* $\mathcal{J} = \mathcal{N} \setminus \mathcal{N}^*$ *of all non-invertible elements of* $\mathcal{N}$ *is a subgroup of* $(\mathcal{N},+)$.

Definition 1.2.4. *A near ring* $\mathcal{N}$ *is known as* **zero-symmetric** *if* $0\cdot u = 0$ *for every* $u \in \mathcal{N}$. *Let* $\mathcal{N}$ *be a zero-symmetric left near ring with* $C(\mathcal{N})$ *as its multiplicative center. For* $v,w \in \mathcal{N}$, *the symbols* $[v,w]$ *and* $v \circ w$ *denote the commutator* $vw - wv$ *and the anti-commutator* $vw + wv$, *respectively.*

Definition 1.2.5. *A near ring* $\mathcal{N}$ *is known as* **2-torsion free** *if*

$2u = 0 \Longrightarrow u = 0$ *for every* $u \in \mathcal{N}$.

Definition 1.2.6. *A near ring* $\mathcal{N}$ *is known as* **3-prime** *if for*

$v,w \in \mathcal{N}, v\mathcal{N}w = 0 \Longrightarrow v = 0$ *or* $w = 0$.

Definition 1.2.7. *An additive subgroup* $\mathcal{J}$ *of a near ring* $\mathcal{N}$ *is known as a* **Jordan ideal** *of* $\mathcal{N}$ *if* $k \circ u \in \mathcal{J}$ *and* $u \circ k \in \mathcal{J}$ *for all* $k \in \mathcal{J}$ *and* $u \in \mathcal{N}$.

Definition 1.2.8. *An additive map* $f : \mathcal{N} \longrightarrow \mathcal{N}$ *is known as commuting on a non empty subset* $\mathcal{S}$ *of a near ring* $\mathcal{N}$ *if* $[f(u),u] = 0$ *for all* $u \in \mathcal{S}$. *A mapping* $d : R \longrightarrow R$ *is known as a* **multiplicative derivation** *on a ring* R *if* $d(vw) = d(v)w + vd(w)$ *for every* $v,w \in R$.

Definition 1.2.9. *[28] A finite ring, R is called a Von Neumann regular ring,* $VNR(R)$ *if and only if for every* $a \in R$, *there exists some* $x \in R^*$ *such that* $a^2x = a$ *where* a *is a regular element and* $a = ue$ *for some* $u \in R^*$ *and* $e \in Idem(R)$, *the idempotent set of* R .

Definition 1.2.10. *[69], Definition 2) Let* $\mathcal{R}$ *be a nonempty set in which there are defined two binary operations called addition and multiplication. For* $a,b \in \mathcal{R}$. *Then* $\mathcal{R}$ *is called a ring if the following axioms hold ;*

(i). For all $a, b \in \mathcal{R}$*,* $(a+b) \in \mathcal{R}$

(ii). For all $a, b, c \in \mathcal{R}$*,* $(a+b)+c = a+(b+c)$

(iii). There exists an element denoted by 0 *such that for all* $a \in \mathcal{R}$*,* $a+0 = 0+a = a$

(iv). For any $a \in \mathcal{R}$ *there exists an element denoted by* $-a \in \mathcal{R}$ *such that* $a + (-a) = (-a) + a = 0$

(v). For all $a, b \in \mathcal{R}$*,* $a+b = b+a$

(vi). For all $a, b \in \mathcal{R}$*,* $ab \in \mathcal{R}$

(vii). For all $a, b, c \in \mathcal{R}$*,* $(ab)c = a(bc) \in \mathcal{R}$

(viii). For all $a, b, c \in \mathcal{R}$*,* $a(b+c) = ab+ac, (a+b)c = ac+bc$

Definition 1.2.11. *([69], Definition 5) Let* $\mathcal{R}$ *be a ring.*

(i). An element $1 \in \mathcal{R}$ *such that* $1a = a1 = a$ *for all* $a \in \mathcal{R}$ *is called an identity element or identity or unity of* $\mathcal{R}$*.*

(ii). Let $a, b \in \mathcal{R}$*. Then* a *and* b *are said to commute if the products* $ab = ba$*. If any two elements of* $\mathcal{R}$ *commute then* $\mathcal{R}$ *is said to be commutative.*

(iii). Let $a, b \in \mathcal{R}$ *and* 0 *be a zero element of* $\mathcal{R}$ *such that* $ab = 0$*. Then* a *and* b *are called zero divisors and if* $a \neq 0$ *and* $b \neq 0$ *then* a *and* b *are called proper divisors of zero.*

Definition 1.2.12. *([69], Definition 6) A commutative ring with an identity* $1 (1 \neq 0)$ *and no proper divisors of zero is called an integral domain.*

Definition 1.2.13. *([45], Definition 1.1) If* $\mathcal{G}$ *is a nonempty set, a binary operation* $*$ *on* G *is a function* $* : G \times G \to G$*.*

Remark 1.2.1. *The binary operation* $*$ *is thought of as either a multiplication* (.) *or addition* $(+)$ *of the elements of* $\mathcal{R}$.

Definition 1.2.14. *([45], Definition 1.7) Two groups* $(G, *)$ *and* (H, o) *are said to be isomorphic if there is a one-to-one correspondence* $\theta : H \to G$ *such that* $\theta(g_1 * g_2) = \theta(g_1 o g_2)$ *for all* $g_1, g_1 \in G$. *The mapping* θ *is called an isomorphism and we say that* G *is isomorphic to* H *and denote it by* $G \cong H$.

Remark 1.2.2. *If* θ *satisfies the above property but is not one-to-one correspondence, we say that* θ *is a homomorphism.*

Definition 1.2.15. *([45], Definition 1.8) A graph is a pair* $\Gamma = (\nu, \varepsilon)$ *where:*

(i). ν *is a finite set of vertices and*

(ii). ε *is the collection of unordered pairs of vertices called edges.*

Definition 1.2.16. *([45], Definition 2.1) A nonempty subset* S *of the group is a subgroup of* G *if* S *is a group under binary operation of* G. *We use the notation* $S \leq G$ *to indicate that* S *is a subgroup of* G.

Definition 1.2.17. *([45], Definition 2.4) The number of elements in a finite* G *is called the order of* G *and is denoted by* $|G|$.

Definition 1.2.18. *([45], Definition 5.4) An ideal* P *of a ring* R *is a prime ideal if whenever* $ab \in P$, *then either* $a \in P$ *or* $b \in P$.

Definition 1.2.19. *([45], Definition 5.1) A subset* I *of a ring* R *is an ideal if:*

(i). $a, b \in I$ *then* $a + b \in I$

(ii). $r \in R$ *and* $a \in I$, *then* $ra \in I$ *and* $ar \in I$. *We write* $I \lhd R$ *and say* I *is an ideal of* R.

Definition 1.2.20. *([45], Definition 5.2) An ideal I that is singularly generated, i.e $I = (a)$, is called a principal ideal.*

Definition 1.2.21. *([45], Definition 1.7) A ring with only principal ideals is called a principal ideal ring (PIR). And similarly, a domain with only principal ideals is a principal ideal domain (PID).*

Definition 1.2.22. *([45], Chapter 5, section 5.1) A field is a commutative ring with unity in which every nonzero element has a multiplicative inverse.*

Definition 1.2.23. *([69], Definition 11) Let $\mathcal{R}$ and $\mathcal{S}$ be rings and let $f : \mathcal{R} \to \mathcal{R}$ be a mapping such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{R}$. Then f is said to be a homomorphism from $\mathcal{R}$ to $\mathcal{S}$.*

Definition 1.2.24. *([69], Definition 12) Let $\mathcal{R}$ and $\mathcal{S}$ be rings and $f : \mathcal{R} \to \mathcal{R}$ be a homomorphism. then $K = \{x \in \mathcal{R} \mid f(x) = O_s\}$ is an ideal of $\mathcal{R}$ which is called the kernel of f, denoted by $ker f$.*

Definition 1.2.25. *([51], Definition 5.21) Let $\mathcal{M}$ be an additively written abelian group and $\mathcal{R}$ a ring. Then $\mathcal{M}$ is said to be a (right) $\mathcal{R}$-module if a law of composition of $\mathcal{M} \times \mathcal{R}$ into $\mathcal{M}$ is defined uniquely as xa for $a \in R$,$x \in M$ with the following properties for $x, y \in M$ and $a, b \in R$:*

(i). $(x + y)a = xa + ya$

(ii). $x(a + b) = xa + xb$

(iii). $x(ab) = (xa)b$

Definition 1.2.26. *([51], Definition 6.1) Let a be an element of the ring $\mathcal{R}$.*

(i). if there exists an element b of $\mathcal{R}$ such that $aob = 0$, a is said to be right-quasi regular and to have b as the right quasi inverse

(ii). if there exists an element c of $\mathcal{R}$ such that $c o a = 0$, a is said to be left-quasi regular and to have c as the right quasi inverse

(iii). an element a is said to be quasi regular if it is both right-quasi regular and left-quasi regular.

Definition 1.2.27. *([51], Definition 6.6) The Jacobson radical, $\mathfrak{J}(\mathcal{R})$ of a ring R is defined as $\mathfrak{J}(\mathcal{R}) = \{a : a \in R, aR \text{ is right} - \text{quasi regular}\}$.*

Definition 1.2.28. *([51], Definition 7.2) Let c be an element of an arbitrary ring $\mathcal{R}$. If there exists an element c' of $\mathcal{R}$ such that $c = cc'c$, c is said to be a regular element of $\mathcal{R}$. The ring $\mathcal{R}$ is said to be a regular ring if each of its elements is regular.*

Definition 1.2.29. *([51], Definition 7.17) An element a of the ring $\mathcal{R}$ is said to be $\mathcal{G}$-regular if $a \in G(a)$. An ideal is said to be $\mathcal{G}$-regular if each of its elements is $\mathcal{G}$-regular.*

Definition 1.2.30. *([51], Definition 7.18) The radical $\mathfrak{R}(\mathcal{R})$ of the ring $\mathcal{R}$ is defined as $\mathfrak{R}(\mathcal{R}) = \{b : b \in \mathcal{R}, b \text{ is } \mathcal{G} - \text{regular}\}$.*

1.3 Statement of the Problem

The problem of classification of algebraic structures has contributed greatly in understanding their structural properties and applications. For instance, Galois showed that the roots of a general quintic polynomial, f cannot be solved using any radical by considering the automorphism groups of the splitting field of f. The group classification problem for both finite and infinite groups is however complete. On the other hand, the ring classification problem is still open. A lot of current studies involving finite rings with identity have independently taken the trajectory of determining and characterizing the algebraic structures associated with the various compartments of the ring R. These studies have left the non-linear aspects of the structures of R fairly untouched. In particular, the generalization of the regular elements of any classes of near-rings is hardly available in literature. This study has therefore determined and classified the regular elements of the zero symmetric local near-ring, $\mathcal{N}$ admitting Frobenius derivation up to isomorphism.

1.4 Objectives of the Study

1.4.1 Main Objective

The main objective of this study was to determine and classify the regular elements and Von-Neumann inverses of the zero-symmetric local near rings with $n-$nilpotent radical of Jordan ideals admitting Frobenius derivations.

1.4.2 Specific Objectives

The specific objectives of this study were:

(i). To determine the structures of classes of zero symmetric local near rings $\mathcal{N}$ with $n-$nilpotent radical of Jordan ideals $J(\mathcal{N})$, for $n = 2, n \geq 3$ for char $\mathcal{N} = p, p^2$ and $p^k : k \geq 3$ convoluted with Frobenius derivations.

(ii). To determine the commutation over $\mathcal{N}$ constructed using $J(\mathcal{N})$ and the Frobenius derivations $d : \mathcal{N} \to \mathcal{N}$ and $d : \mathcal{N} \to J(\mathcal{N})$.

(iii). To characterize and classify $\mathcal{N}, R(\mathcal{N}), \Gamma(\mathcal{N})$ and the inverses of $\mathcal{N}$.

1.5 Research Methods

1.5.1 Introduction

In the sequel, we give some techniques, characterization procedures and theorems that have been used to achieve our objectives.

1.5.2 Idealization Method

Suppose that $\mathcal{R}$ is a commutative ring with 1, and $\mathcal{M}$ is a unitary $\mathcal{R}$-module. Then we call the set $R + M$ supplied with coordinatewise $+$ and (.), the idealization of $\mathcal{M}$. Using Galois near-rings R_0 with R_0-modules $\mathcal{M}$, we have used the idealization method to:

(i). reduce the results concerning R_0 to the ideal case.

(ii). generalize results from R_0 to the R_0-modules.

(iii). construct the new classes of finite zero symmetric local near-rings of char p, p^2 and $p^k : k \geq 3$.

1.5.3 Commutativity of Near-rings

Commutative properties of near rings allow them to admit morphisms with a host of properties. We have applied the following theorems due to Asma and Inzamam [8] to determine commutation properties of the near ring N in question via $J(N)$ and the Frobenius derivation as the morphism.

Theorem 1.5.1. *([8], Theorem 1) Let J be a non-zero Jordan ideal of a 2-torsion free 3-prime near-ring N. If d_1, d_2 are two nonzero derivations on N such that d_2 is commuting on J and and $[d_1(u), d_2(k)] = [u, k]$ for all $k \in J$ and $u \in N$, then either $d_1 = 0$ on J or N is a commutative ring.*

Theorem 1.5.2. *([8], Theorem 2) Let J be a non-zero Jordan ideal of a 2-torsion free 3-prime near-ring N. If d is a nonzero derivation on N satisfying $d([k, u]) = [d(k), u]$ for all $k \in J$ and $u \in N$, then N is a commutative ring.*

Theorem 1.5.3. *([8], Theorem 3) Let J be a non-zero Jordan ideal of a 2-torsion free 3-prime near-ring N. If d is a nonzero derivation on N satisfying $[d(u), k] =$*

$[u,k]$ for all $k \in J$ and $u \in N$, then either the elements of J commute under the multiplication of N or N is a commutative ring.

Theorem 1.5.4. *([8], Theorem 4) Let J be a non-zero Jordan ideal of a 2-torsion free 3-prime near-ring N. If d is a derivation on N satisfying $d([k,u]) = d(k) \circ u$ for all $k \in J$ and $u \in N$, then either $d = 0$ or the elements of J commute under the multiplication of N.*

1.5.4 Raghavendran's Characterization Procedure and The Gap Package SONATA

As a supplement to idealization, Raghavendran's characterization procedure [62] was used in our study to determine the structures of $R(\mathcal{N}) = \mathcal{N}^* \cup \{0\}$

Theorem 1.5.5. *([62], Theorem 2) Let R be a finite ring with multiplicative identity $1 \neq 0$ whose zero divisors form an additive group $Z(R)$. Then,*

(i). $Z(R)$ is the Jacobson radical of R.

(ii). $|R| = p^{nr}$; and $|Z(R)| = p^{(n-1)r}$ for some prime integer p and some positive integers n and r.

(iii). $(Z(R))^n = 0$.

(iv). the characteristic of the ring R is p^k for some positive integer k with $1 \leq k \leq n$; and

(v). if the characteristic is p^n, then R is commutative.

Theorem 1.5.6. *([62], Theorem 8) Let R, p, r be as in Theorem 1.5.5. Then*

(i). R will contain a sub-ring isomorphic to $GR(p^{kr}, p^k)$ if, and only if the characteristic of R is p^k, and

(ii). if R_2, R_3 are any two sub-rings of R, both isomorphic to $GR(p^{kr}, p^k)$, there will be an invertible element a in R such that $R_2 = a^{-1}.R_3.a$.

$R(\mathcal{N})$ were validated using the Gap package SONATA, a software package which give algorithms for the construction and analysis of finite near-rings.

1.6 Significance of the Study

Near rings are generalized rings. On comparing with the standard class of rings, endomorphism rings of abelian groups, it can be seen that ring theory describes a "linear theory of group mappings,"while near rings deal with the general "non-linear theory." A great number of linear results have been transferred to the general nonlinear case with some suitable changes. The results of near rings can be used in various fields inside and outside of pure mathematics [59]. Efficient codes and block designs can be constructed with the help of finite near rings. In mathematics, there are applications of near ring theory in functional analysis, algebraic topology, and category theory. Near rings also find applications in digital computing, automata theory, sequential mechanics, and combinatorics (see [58] and the references therein).

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In this chapter, we provide a detailed survey on the literature regarding the near rings with generalized derivations, regular elements and Von Neumann inverses and zero symmetric near rings with Jordan ideals admitting Frobenius derivations with an aim of depicting the gaps which our study has addressed.

2.2 Near Rings with Generalized Derivations

Many results in literature have demonstrated how the global structure of a near-ring $\mathcal{N}$ is often tightly connected to the behavior of additive mappings defined on $\mathcal{N}$. For example, Sammana et al [65] attempted to give an expositional study on near-rings with generalized derivations and established the rich algebraic interplay between generalized derivations and commuting structures of $\mathcal{N}$. However, in some cases, an arbitrary derivation d was used instead of the generalized one. This is against the convention which requires that the type and nature of a differential identity in question should be specified. According to [3], an additive mapping $D : \mathcal{N} \mapsto \mathcal{N}$ is said to be a right (resp., left) generalized derivation with associated derivation d if $D(xy) = D(x)y + xd(y)$ or $D(xy) = d(x)y + xD(y)$, for all $x, y \in \mathcal{N}$, and D is said to be a generalized derivation with associated derivation d on D if it is both a right and a left generalized derivation on $\mathcal{N}$ with associated derivation, d. Every derivation on $\mathcal{N}$ is a generalized derivation. Familiar examples of generalized derivations are the generalized inner derivations and derivations incorporating left multiplier, that is, an additive mapping $D : \mathcal{N} \mapsto \mathcal{N}$ satisfying $D(xy) = D(x)y$

for all $x, y \in \mathcal{N}$. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view with a specific focus on local near-rings might be interesting

The results of [2] coined a definition for the term IFP ideal and considered the relations between prime ideals and IFP-ideals. Furthermore, it was proved that a right permutable or left permutable equiprime near-ring has no non-zero nilpotent elements.

Proposition 2.2.1. *([2], Proposition 3.2)*

If $\mathcal{N}$ is a right (or left) permutable 3-prime near-ring, then $\mathcal{N}$ has no non-zero nilpotent elements.

Proposition 2.2.2. *([2], Proposition 3.4) If p is an IFP-ideal and a 3-(semi) prime ideal of $\mathcal{N}$, then p is a completely (semi) prime ideal.*

The two propositions apply permutability of a near-ring and 3-primeness respectively to determine the existence of non-zero nilpotent elements and to determine whether the near-ring is completely prime near-ring. Our study does not involve permutable 3-prime or prime ideal of IFP.

Proposition 2.2.3. *([2], Proposition 3.8)*

Let $\mathcal{N}$ be a medial near-ring and p a 3-prime ideal of $\mathcal{N}$. Then P is an IFP-ideal.

It is clear that this proposition addressed a non-local near-ring and also omitted the discussion of Jordan ideal of $\mathcal{N}$. The matter in [73] is about behaviour of homomorphisms and derivations on certain rings especially prime rings.

Theorem 2.2.1. *([73], Theorem 3.1) Let R be a 2-torsion-free prime ring and let J be a Jordan ideal and a subring of ring R. If θ is an automorphism of R and $\delta : R \to R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $U \epsilon J$, then either $J \subseteq Z(R)$ or $\delta(J) = (0)$.*

This theorem proves that the existence of a Jordan left derivation on a Lie ideal U of 2-torsion-free prime ring R is possible only if $U \subseteq Z(R)$ or $\delta(U) = (0)$. However, this deviates from our study in a number of points. This result is on 2-torsion-free prime rings while ours deal with zero-symmetric local near-rings. Secondly, this result was concerned with Jordan left derivations on Lie ideals whilst our study is fixed on Jordan ideals which admit Frobenius derivation instead of left derivation.

Theorem 2.2.2. *([73], Theorem 4.1) Let R be a prime ring, I a non zero right ideal of R and let (θ, ϕ) be automorphisms of R. Suppose that $\delta : R \to R$ is a (θ, ϕ)-derivation of R.*

(i). If δ acts as a homomorphism on I, then $\delta = 0$ on R.

(ii). If δ acts as an antihomomorphism on I, then $\delta = 0$ on R.

Theorem 2.2.3. *([73], Theorem 4.2) Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal and a subring of R. Suppose that θ is an automorphism of R and $\delta : R \to R$ is a left (θ, θ)-derivation on R.*

(i). If δ acts as a homomorphism on J, then $\delta = 0$ on R.

(ii). If δ acts as an antihomomorphism on J, then $\delta = 0$ on R.

These two theorems above focused on the behaviour of left derivation, especially as homomorphisms or antihomomorphisms. Yet, still this deviates from our study which focuses on the behaviour of elements of the near-ring and not on the maps acting on the ideals of the near-ring. Results of [72] answered an open question in the theory of minimal ideals in near-rings to the negative, that the heart of a zero-symmetric sub-directly irreducible near-ring is subdirectly irreducible again.

Proposition 2.2.4. *([72], Proposition 2.1) Let $\mathcal{N}$ be a non-zero symmetric near-ring. Let L be a minimal left ideal such that L satisfies the DCC on $\mathcal{N}$-subgroups* *contained in L. Suppose $\mathcal{M} \subseteq L$ is an $\mathcal{N}$-subgroup such that $M \neq L$. Then L and M cannot be $\mathcal{N}$-isomorphic.*

This result focused on minimal left ideals L which do not properly contain $\mathcal{N}$-subgroups which are $\mathcal{N}$-isomorphic to L. The ideals considered in our study are Jordan ideals which admit Frobenius derivations.

Theorem 2.2.4. *([72], Theorem 3.1) Let N be a zero symmetric near-ring with DCCN and I a minimal ideal. Then I is isomorphic to a finite direct sum of minimal left ideals of the near-ring N, all of the summands being $\mathcal{N}$-isomorphic. I contains a minimal left ideal L such that $L^2 \neq \{0\}$.*

This theorem analyses the isomorphic properties of the minimal ideal of a near-ring as opposed to the Jordan ideals considered in our study. In [8], the commutative properties of a 3-prime near-ring with some differential identities on Jordan ideals were investigated.

Theorem 2.2.5. *([8], Theorem 1) Let J be a nonzero Jordan ideal of a 2-torsion free 3-prime near-ring $\mathcal{N}$. If d_1, d_2 are two nonzero derivations on $\mathcal{N}$ such that d_2 is commuting on J and $[d_1(u), d_2(k)] = [u, k]$ for all $k \in J$ and $u \in \mathcal{N}$, then either $d_1 = 0$ on J or N is a commutative ring.*

This result clearly, investigated the condition necessary for a 2-torsion-free prime ring which admits strong commutativity preserving derivation to be commutative. This result does not consider the Jordan ideal and the derivations considered are not of the Frobenius type. This assertion is reinforced by the following theorems.

Theorem 2.2.6. *([8], Theorem 2) Let J be a non zero Jordan ideal of a 2-torsion free 3-prime near ring N. If d is a non zero derivation on $\mathcal{N}$ satisfying $d([k, u]) = [d(k), u]$ for all $k \in J$ and $u \in N$,then $\mathcal{N}$ is a commutative ring.*

Theorem 2.2.7. *([8], Theorem 3) Let J be a non zero Jordan ideal of a 2-torsion free 3-prime near-ring $\mathcal{N}$. If d is a non zero derivation on $\mathcal{N}$ satisfying $[d(u), k] = [u, k]$ for all $k \in J$ and $u \in \mathcal{N}$, then either the elements of J commute under the multiplication of $\mathcal{N}$ or $\mathcal{N}$ is a commutative ring.*

Theorem 2.2.8. *([8], Theorem 4) Let J be a non-zero Jordan ideal of a 2-torsion free 3-prime near-ring $\mathcal{N}$. If d is a derivation on $\mathcal{N}$ satisfying $([k, u]) = d(k) \circ u$ for all $k \in J$ and $u \in \mathcal{N}$, then either $d = 0$ or the elements of J commute under the multiplication of $\mathcal{N}$.*

It is also clear that the near-ring considered in [8] is not the zero-symmetric local near-ring. Also the non zero derivations considered may or may not be Frobenius.

2.3 Regular Elements and Von-Neumann Inverses

Osba, Henricksen and Osama [56] conducted a classification survey on combining local and Von Neumann regular rings as a basis upon which the regularity properties of rings and their ideals could be explored. Despite the fact that the rings studied in [56] were finite and their Von Neumann inverses gave asymptotic patterns, the study gave a baseline analysis regarding only the elements of $\mathbb{Z}_n$ which have symmetrical Von-Neumann inverses. The structural characterization did not consider cases of idealization. Their findings demonstrated how to combine the Von Neumann inverses of classes of rings such as the power series rings and the ring of integers. They however did not count the number of regular elements in a given finite ring nor did they give the structural formulae for the regular elements and the Von Neumann inverses of the specified classes of rings.

In a separate but related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [55] where they characterized the regular elements in $\mathbb{Z}_n$, the ring of integers modulo n. They studied the arithmetic functions denoted as $V(n)$ and determined the relationship between $V(n)$ and the Euler's phi function, $\varphi(n)$. This gave an extension of the ring theoretic algebra employed in counting the regular elements of $\mathbb{Z}_n$ to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in $\mathbb{Z}_n$, then $a^{(-1)} \equiv a^{\varphi(n)-1}$ $(mod\ n)$. They developed a criterion for getting the possible Von Neumann inverses in the set of regular elements of $\mathbb{Z}_n$ and furthermore explored the asymptotic properties of $V(n)$. Their research however concentrated on direct products of local rings constructed from the trivial Galois ring $\mathbb{Z}_n$ for n, where n is a product of primes qualifying $GR(P^{\alpha i})$ to be the typical ring. One of the fundamental results they obtained for such products of local rings is as follows;

Theorem 2.3.1. *([55], Section 3) If* $R = \prod_{i=1}^{m} R_i = \underbrace{R_1 \times R_2 \times ... \times R_m}_{m-tuple}$, *then the number of Von Neumann regular elements counting multiplicities is*

$$|V_r(R)| = \prod_{i=1}^{m}(|R_i| - |M_i| + 1)$$

where R_i *- local,* M_i *- maximal ideal of* R_i *and*

$$\varphi(n) = \prod_{i=1}^{m}(P_i^{\alpha i} - P_i^{\alpha i-1}) = n\prod_{p\backslash n}(1 - \tfrac{1}{p}).$$

From the results above, it would be interesting to extend their ideas to classes of local rings expressed as direct summands of Galois rings and by extension to the dual results in local near-rings. Studies on rings have led to the analysis of regular elements with involution $*$. In [39], a number of results exist on properties of such elements. For instance;

Theorem 2.3.2. *([39], Theorem 1) Let* R *be a primitive ring with* $*$ *and suppose that* $a \neq 0$ *in* S *is such that* $ak_0a = 0$. *Then* R *contains a minimal right ideal* ρ *such that the commuting ring of* R *on* ρ *is a field. Moreover, if* R *is a simple ring with unit element, then* R *is isomorphic to* F_n, *the set of* $n \times n$ *matrices over the field* F, *for some* n.

Thus, this theorem focused on primitive ring with an involution. It also showed that the condition for a ring to be a field is that it must contain a minimal right ideal. In our study, on the contrary, the research has considered local near-rings with Jordan ideals.

Theorem 2.3.3. *([39], Theorem 2) Let R be a ring having no non trivial ideals invariant with respect to $*$. If all non zero elements of k_0 are invertible in R, then R is a division ring, the direct sum of a division ring and its opposite, or F_2, the 2×2 matrices over a field F.*

This result considers on division rings with ideals which are invariant with respect to the involution $*$. On the other hand, our study focuses on the elements of the local near-rings and on Jordan ideals in such near-rings.

Theorem 2.3.4. *([39], Theorem 4) Let R be a prime ring with involution in which the non-zero elements of k_0 are regular. Then R is either a domain or an order in the 2×2 matrices over a field.*

This result is similar to that of our study, although our study has not considered involutions. Furthermore, this result does not consider ideals of any type, although it considers the derivations induced by involutions. David and Badawi [28] studied the Von Neumann regular and related elements in commutative rings. In particular, they characterized the Von Neumann regular elements of R by determining the idempotent elements, the π-regular elements, the Von Neumann local elements and the clean elements of R. They also investigated the subgraphs of the zero-divisor graph, $\Gamma(R)$ of R induced by the above elements. Among the results obtained in their study is that;

Theorem 2.3.5. *([28], Theorem 3.7) Let R be a commutative ring and M an R-module,*

(i). $VNR(R \oplus M) = \{(r, rm) \mid r \in VNR(R),\ m \in M\}$.

(ii). *$R \oplus M$ is Von Neumann Regular if and only if R is Von Neumann regular and $M = \{0\}$.*

(iii). $VNL(R \oplus M) = \{(r, rm) \mid r \in VNR(R), m \in M\} \cup \{(1 + r, rm) \mid r \in VNR(R), m \in M\}$.

(iv). *R is a Von Neumann Local Ring when $R \oplus M$ is a Von Neumann Local Ring.*

(v). *Suppose that there is $m \in M$ with $ann_{(R)}(m) = \{0\}$. Then $R \oplus M$ is a Von Neumann Local Ring if and only if R is a Von Neumann Local Ring with $idem(R) = \{0, 1\}$.*

(vi). *If M is a ring extension of R, then $R \oplus M$ is a Von Neumann Local Ring if and only if R is a Von Neumann Local Ring with $idem(R) = \{0, 1\}$.*

The relationship between the graphs of idempotent elements, Von Neumann regular elements, π-regular elements, the clean regular elements and the generalized Beck's graph, $\Gamma(R)$ was obtained as follows;
$\Gamma(Idem(R)) \subseteq \Gamma(VNR(R)) \subseteq \Gamma(\pi{-}R(R)) \subseteq \Gamma(cln(R)) \subseteq \Gamma(R)$ and $\Gamma(VNR(R)) \subseteq \Gamma(VNL(R)) \subseteq \Gamma(cln(R))$ for any commutative ring R. The results of the study were not extended to the graph theoretic properties of the $\Gamma(R^*)$ and hence by extension not addressed to the graphs of the units of the near-rings.
The most recent breakthrough on a characterization of Regular elements of idealized finite rings can be seen in Owino and Musoga [52] where they studied the Regular Elements of a Class of Commutative Completely Primary Finite Rings, $CPFRs$. Their characterization was based on the construction and structural classification of the Von Neumann Regular elements starting with the Galois rings as a basis upon which the structures and orders of $VNR(R)$ for other $CPFRs$ were characterized. Intuitively, the methods employed in Owino and Musoga [52] were

similar to the ones used by Osama and Emad [55] although they were working on completely distinct classes of rings. The object motivating this study is an endeavour to provide dual results in near-ring set up in order to generalize the notion of regular elements $R(\mathcal{N})$ whose algebraic structure is an abelian group.

2.4 Zero-Symmetric Local Near-Rings

Sheaf representations of commutative rings and reduced rings can be unified using symmetric rings. In depth study of symmetric rings, discussion of basic examples and extensions have been done in [23]. It has been shown that a non-reduced symmetric ring can be constructed from a reduced ring.

Theorem 2.4.1. *([23], Theorem 2.3) Let R be a ring and n any positive integer. If R is reduced, $R[x]/(x^n)$ is a symmetric ring, where (x^n) is the ideal generated by x^n .*

This theorem proves the existence of a symmetric ring that is neither commutative nor reduced. It also shows that the class of symmetric rings contains both commutative and reduced rings. In our case, we have considered local near-rings which may or may not be rings at all. The ideas of 2-primality of a ring and ability of a ring to satisfy a polynomial identity with coefficients in the ring of integers were related by [23] in the following proposition.

Proposition 2.4.1. *([23], Proposition 2.7) Let R be a ring.*

(i). R is reduced if and only if R is semi prime and symmetric, if and only if R is semi prime and IFP, if and only if R is semi prime and 2-primal.

(ii). R is a domain if and only if R is prime and reduced, if and only if R is prime and symmetric, if and only if R is prime and IFP if and only if R is prime and 2-primal.

(iii). Let R be a semi prime ring such that each non-zero right ideal contains a non-zero ideal. Then R is reduced.

(iv). let R be a p1-ring. Then R is a semi prime ring such that each non-zero right ideal contains a non-zero ideal if and only if R is reduced.

It is observable that this result focused majorly on reduced and non-reduced rings as opposed to local near-rings with ideals which is the centre of our discussion. Furthermore, the rings considered in [23], were not the ones containing Jordan ideals. The following proposition focused on the symmetry of the rings.

Proposition 2.4.2. *([23], Proposition 3.3) Let R be a ring and suppose that $Z(R)$ contains an infinite subring every non-zero elements of which is regular in R. Then R is symmetric if and only if $R[x]$ is symmetric if and only if $R[x : x^{-1}]$ is symmetric.*

Thus this proposition considered regular elements of a ring, but omitted the Von-Neumann inverses of such elements.

Proposition 2.4.3. *([23], Proposition 3.6)*

(1). Let R be a ring and I be a proper ideal of R. If $R|I$ is symmetric and I is reduced, then R is symmetric.

(2). For an abelian ring R, R is symmetric if and only if eR and $(1-e)R$ are symmetric for every idempotent e and R.

This result touched on proper ideals of a ring and not Jordan ideals. Classes of near-rings which satisfy the polynomial identities below were considered by [1];

(i). For each x, y in a near-ring N, there exist positive integers $t = (x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm y^s x^t$.

(ii). For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm x^t y^s$.

Theorem 2.4.2. *([1], Theorem 2.1) Suppose that N is a near-ring which satisfies (i) above and the idempotent elements of N are multiplicative central. Then the set A of all nilpotent elements of N is a subnear-ring with trivial multiplication, and the set B of all idempotent elements of N is a subnear-ring with $(B, +)$ abelian. Furthermore, $N = A \oplus B$.*

This theorem focused on idempotent elements of subnear-rings instead of regular elements and their Von Neumann inverses on the near-rings which is at the centre of our study. Similarly, the following theorem also focused on zero-commutative near-ring instead of zero-symmetric near-ring algebra.

Theorem 2.4.3. *([1], Theorem 2.2) Let N be a zero-commutative near-ring which satisfies condition (ii) above and the idempotent elements of N are multiplicative central. Then the set A of all nilpotent elements of N is a subnear-ring with trivial multiplication, and the set B of all idempotent elements of N is a subnear-ring with $(B, +)$ abelian. Furthermore, $N = A \oplus B$.*

Further studies on near-rings by [1] showed that there are certain near-rings which are actually rings.

Theorem 2.4.4. *([1], Theorem 3.1) Let N be a d.g near-ring which satisfy (i). Then $\mathcal{N}$ is commutative.*

Theorem 2.4.5. *([1], Theorem 3.2) Let N be a d.g near-ring satisfying (ii). Then $\mathcal{N}$ is commutative.*

The two theorems point to the fact that the conditions (i) and (ii) are necessary for a near-ring to be a commutative ring. It is clear that this study is not concerned with local near-rings with Jordan ideals, as in the case of our study.

The following two propositions considers zero-symmetric near-rings which contain minimal ideals which are not similar to Jordan ideals. Furthermore, they involve non-local near-rings which are different from the ones we have considered.

Proposition 2.4.4. *([72], Proposition 5.1) Let $\mathcal{N}$ be a zero symmetric near-ring containing a minimal ideal H. Suppose that $\mathcal{N}$ is 0-primitive on the $\mathcal{N}$-group Γ. Then $\mathcal{N}$ is sub-directly irreducible near-ring with heart in H.*

The condition required of a zero symmetric near-ring with a generator to be a ring is considered by [72] in the following theorem.

Theorem 2.4.6. *([72], Theorem 5.3) Let $\mathcal{N}$ be a zero symmetric near-ring which is 0-primitive on Γ. We assume that $\mathcal{N}$ is not a ring. Suppose that $I := (0 : \Theta_0) \neq \{0\}$ and there is a finite number n of elements $r_1, ..., r_n \subseteq \Theta$ such that $\bigcap_{i=1}^{n}((0 : r_i) \cap I) = \{0\}$. Then $I = (0 : \Theta_0)$ is the unique minimal ideal of $\mathcal{N}$. In particular, $(0 : \Theta_0)$ contains a right identity element I_r and a direct summand as a left ideal of $\mathcal{N}$ and $\mathcal{N} = (0 : \Theta_1) + (0 : \Theta_0)$ with $J_{1/2}(\mathcal{N}) \subseteq (0 : \Theta_1)$. If there is another $\mathcal{N}$-group Γ,on which $\mathcal{N}$ acts 0-primitively also, then $\Gamma \simeq_{\mathcal{N}} \Gamma_1$.*

The article [46] considered properties of rings with involution. In particular, the structure of 2-torsion-free rings with involution whose non-zero symmetric elements do not annihilate each other.

Theorem 2.4.7. *([46], Theorem 3) Suppose S' has no zero divisors. Then R has a unique maximal nilpotent ideal $\mathcal{N}$ satisfying:*

(i). $\mathcal{N} \subset K$

(ii). $\mathcal{N}^3 = 0$

(iii). If $x \epsilon \mathcal{N}$, *then* $x^2 = 0$

(iv). $\mathcal{N}$ *contains all nil one-sided ideals of R.*

(v). $R|\mathcal{N}$ is a 2-torsion-free ring with involution containing no nil ideals.

(vi). If $S(R|\mathcal{N})$ are the symmetric elements of $R|\mathcal{N}$, then $S'(R|\mathcal{N})$ has no zero divisors.

This result deviates from our study in that whereas it focused on ideals of a ring, our study involved ideals of a near-ring which can be a ring or not. Furthermore, the ideals considered in this result were those of nilpotent nature but the ones considered in our study are specifically Jordan in nature.
In [27] a generalization of near-rings is done for the case where the additive structure is not necessarily associative. Locality of such near-rings were introduced and algorithm for detection of locality generated. Such generalizations were used to coin the definition of loop near-rings whose properties are studied thus;

Proposition 2.4.5. *([27], Proposition 1.7) Let $\mathcal{N}$ and $\mathcal{M}$ be loop near rings and $G = nGm$ an $(\mathcal{N}$,$\mathcal{M})$-bimodule. The following assertions hold: [(i).] $K \subseteq G$ is a left N-submodule if and only if K is a normal subloop in (G,+) and n(a+k)+K=na+K hold for all $k \in \mathcal{N}$ [(ii).] $K \subseteq G$ is a right M-submodule if and only if K is a normal subloop in (G,+) and $MK \subseteq K$ [(iii).] $I \subseteq G$ is a left N-subloop if and only if I is a subloop in (G,+) and $NI \subseteq I$ [(iv). $I \subseteq G$ is a right M-subloop if and only if I is a subloop in (G,+) and $IM \subseteq I$ [(v).] If N is zero-symmetric, then every left N-submodule in G is also an N-subloop.*

In this case the author [27] majored on loop homomorphisms to analyse element-by-element computations. It was also shown that lack of symmetry between the element-wise characterizations of left N-modules and right M-modules are due to absence of left distributivity. In our case though, we have not considered the loop near-rings although we have considered homomorphisms similar to the ones applied in [27]. A deeper study on homomorphisms on loop near rings [], lead the author to consider isomorphisms generated by the homomorphisms in question. For instance given that $\phi : G \mapsto H$ is a homomorphism of some (N, M)-bimodule, then this homomorphism ϕ induces an isomorphism an isomorphism say, $\overline{\phi}$. Furthemore, [] showed that such induced isomorphism determines a bijective correspondence between subloops or submodules of the module in question. Our study on the contrary, does not consider isomorphisms related to loop near rings. In [70], the concept of sandwich near-rings and that of centralizer near-rings were combined to get a classification of zero symmetric 1-primitive near-rings. Such near-rings obtained were dense subnear-rings of centralizer near-rings with sandwich multiplication. The results [70] indeed generalize density theorem for zero symmetric 2-primitive near-rings with identity to much bigger class of zero symmetric 1-primitive near-rings with an identity.

Theorem 2.4.8. *([70], Theorem 4.3) Let $\mathcal{N}$ be a zero symmetric near-ring which is not a ring. Then the following are equivalent:*

(i). $\mathcal{N}$ is 1-primitive

(ii). There exists

(a). a group $(\Gamma, +)$,

(b). a set $X = \{0\} \cup X, \subseteq \Gamma, X \neq \emptyset, 0 \notin X$ and 0 being the zero of Γ.

(c). $S \leq Aut(\Gamma, +)$, with $S(x) \leq X$ and S acting without fixed points on X_1

(d). a function $\emptyset : \Gamma \longrightarrow X$ with $\emptyset|_x = id, \emptyset(0) = 0$ and such that $\forall \nu \in \Gamma$ $\forall s \in S : \emptyset(s(\nu)) = s(\emptyset(\nu))$, such that $\mathcal{N}$ is isomorphic to a dense subnear-ring $\mathcal{M}_s$ of $\mathcal{M}_0(X, \Gamma, \emptyset, S)$ where $X, \Gamma, \emptyset, S$ additionally satisfy the following property (p) :

The results of [70], heavily relied on equivalence relation and the annihilation property of the near rings. In our study, on the contrary, w have not used annihilators. Furthermore, we have not considered such dense sub-near rings. From the foregoing studies, it's worth noting that little has been done regarding a characterization of near-rings via regular elements, Von Neumann inverses and their graphs. In particular, local near-rings admitting Frobenius derivations have attracted not much attention despite the fact that all rings are known to be near-rings and further that commutation criteria on rings and near-rings can be canvassed through derivations on either maximal, prime or Jordan ideals.

CHAPTER THREE

ZERO SYMMETRIC LOCAL NEAR-RINGS OF CONSTRUCTION I

3.1 Introduction

In this chapter, we have used the elements finite dimensional near-module to construct zero symmetric local near-ring of $charp^k : k = 1, 2$. Furthermore, we have characterized the classes of the near-rings constructed and also investigated the commutation over the constructed near-rings using the properties of the Jordan ideal, $J(\mathcal{N})$ and Frobenius derivation d, d_1 and d_2 admitted by $\mathcal{N}$.

3.2 The Construction

Let $R_0 = G\mathcal{N}(p^{kr}, p^k)$ be a Galois near-ring of order p^{kr} and characteristic p^k and let $\mathcal{M} = \langle u_i \rangle : \ i = 1, \cdots h$ be an $h-$dimensional near-module of R_0 so that the ordered pair $(\mathcal{N}, +) = (R_0 \oplus \mathcal{M}, +)$ is a group. On $\mathcal{N}$, let

$$p^k u_i \ = \ \prod_{i=1}^{k} u_i = 0$$

and $u_i r_0 = (r_0)^{d_i} u_i$ when $k = 1, \ 2$ where $r_0 \in R_0, \ k, r$ are invariants and d_i a Frobenius derivation associated with elements of $\mathcal{M}$ and given by; $d_i(u_i) = (u_i)^p$. Let $\mathcal{J}$ be a near-ideal of $\mathcal{M}$ satisfying the condition that whenever $u_i, u_j \in \mathcal{J}$, we have $u_i \circ u_j \in \mathcal{J}$ or $u_i \circ u_j = 0$. If λ_i are any units of R_0, then we can see that the elements of $\mathcal{N} = R_0 \oplus \mathcal{M}$ are of the form: $x = r_0 + \sum_{i=1}^{h} \lambda_i u_i$. In fact, if $x = r_0 + \sum_{i=1}^{h} \alpha_i u_i$ and $y = s_0 + \sum_{i=1}^{h} \beta_i u_i$ are any two elements of $\mathcal{N}$, then we have their product as:

$$x \cdot y \quad = \quad \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big) \cdot \Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big)$$

$$= r_0 s_0 + \sum_{i=1}^{h} \Big\{\beta_i (r_0 + p^k R_0)^{d_i} + \alpha_i (s_0 + p^k R_0)^{d_i}\Big\} u_i. \tag{3.2.1}$$

Theorem 3.2.1. *The triplet, $(\mathcal{N}, +, \cdot)$ with the product given in construction (3.2.1) is a left (respective right) local near-ring.*

Proof. Since $(\mathcal{N}, +)$ is a group, we only show that $(\mathcal{N}, \cdot)$ is a semi group and that the left(right) distributive law holds on $\mathcal{N}$. Let $x, y, z \in \mathcal{N}$ be defined by: $x = r_0 + \sum_{i=1}^{h} \alpha_i u_i$, $y = s_0 + \sum_{i=1}^{h} \beta_i u_i$, $z = k_0 + \sum_{i=1}^{h} \gamma_i u_i$ where $\alpha, \beta, \gamma \in R_0^*$ or $\alpha_i, \beta_i, \gamma_i \in p^k R_0$, then,

$\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\{(s_0 + \sum_{i=1}^{h} \beta_i u_i)(k_0 + \sum_{i=1}^{h} \gamma_i u_i)\Big\}$

$$\begin{aligned}
&= \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(\Big\{s_0 k_0 + \sum_{i=1}^{h} \Big(\gamma_i (s_0 + pR_0)^{d_i} + \beta_i (k_0 + pR_0)^{d_i}\Big)\Big\} u_i\Big) \\
&= r_0 s_0 k_0 + r_0 \Big(\sum_{i=1}^{h} \Big\{\gamma_i (s_0 + pR_0)^{d_i} + \beta_i (k_0 + pR_0)^{d_i}\Big\} u_i\Big) \\
&+ \sum_{i=1}^{h} \alpha_i u_i \Big(s_0 k_0 + \sum_{i=1}^{h} \Big\{\gamma_i (s_0 + pR_0)^{d_i} + \beta_i (k_0 + pR_0)^{d_i}\Big\}\Big) \\
&= r_0 s_0 k_0 + \sum_{i=1}^{h} \Big\{\gamma_i (r_0 + pR_0)^{d_i} + \alpha_i (s_0 + pR_0)^{d_i} + \beta_i (k_0 + pR_0)^{d_i}\Big\} u_i \\
&= r_0 s_0 + \sum_{i=1}^{h} \Big\{\beta_i (r_0 + pR_0)^{d_i} + \alpha_i (s_0 + pR_0)^{d_i}\Big\} u_i (k_0 + \sum_{i=1}^{h} (\gamma_i u_i) \\
&= \Big((r_0 + \sum_{i=1}^{h} \alpha_i u_i)(s_0 + \sum_{i=1}^{h} \beta_i u_i)\Big)\Big(k_0 + \sum_{i=1}^{h} \gamma_i u_i\Big).
\end{aligned}$$

Thus, the multiplication given by the construction is associative.

Next, $\left(r_0 + \sum_{i=1}^h \alpha_i u_i\right)\left\{(s_0 + \sum_{i=1}^h \beta_i u_i) + (k_0 + \sum_{i=1}^h \gamma_i u_i)\right\}$

$$\begin{aligned}
&= \left(r_0 + \sum_{i=1}^h \alpha_i u_i\right)\left\{(s_0 + k_0) + \sum_{i=1}^h (\beta_i + \gamma_i) u_i)\right\} \\
&= r_0 s_0 + r_0 k_0 + \sum_{i=1}^h \left\{(r_o + pR_0)\beta_i + \alpha_i (s_0 + pR_0)^{d_i}\right\} u_i \\
&+ \sum_{i=1}^h \left\{(r_o + pR_0)\gamma_i + \alpha_i (k_0 + pR_0)^{d_i}\right\} u_i \\
&= r_0 s_0 + \sum_{i=1}^h \left\{(r_0 + pR_0)\beta_i + \alpha_i (s_0 + pR_0)^{d_i}\right\} u_i + r_0 k_0 \\
&+ \sum_{i=1}^h \left\{(r_0 + pR_0)\gamma_i + \alpha_i (k_0 + pR_0)^{d_i}\right\} u_i \\
&= \left(r_0 + \sum_{i=1}^h \alpha_i u_i\right) \cdot \left(s_0 + \sum_{i=1}^h \beta_i u_i\right) + \left(r_0 + \sum_{i=1}^h \alpha_i u_i\right) \cdot \left(k_0 + \sum_{i=1}^h \gamma_i u_i\right),
\end{aligned}$$

the desired left distributive law. □

Remark 3.2.1. *Whenever* $d_i = i_{\mathcal{N}}$ *the identity map, then,* $\mathcal{N} = R \oplus \mathcal{M}$ *is a commutative near-ring with identity* $(1, 0, ..., 0)$

Proposition 3.2.1. *The Frobenius derivation* $d : \mathcal{N} \to \mathcal{N}$ *is an endomorphism whenever* $u^{1-p} + v^{1-p} = 1$ *for any* $u, v \in \mathcal{N}$.

Proof. Let $\mathcal{N}$ constructed be a local near-ring with unity, then by Bezout's Theorem, for some non-zero divisors $u, v \in \mathcal{N}$, $u^{1-p} + v^{1-p} = 1$ holds. Now, from the construction, $R_0 = G\mathcal{N}(p^k, p^k)$ is a maximal subset of $\mathcal{N}$, the characteristic of R_0 coincides with the characteristic of $\mathcal{N}$, thus $p^k u = 0,\ k = 1, 2\ \forall\ u \in \mathcal{N}$. Let $u = r_0 + \sum_{i=1}^h \alpha_i u_i$, and $v = s_0 + \sum_{i=1}^h \beta_i u_i$, then clearly u, v are in $\mathcal{N}$ and by definition of our derivation; $d(u) = u^p$. But,

$$\begin{aligned} d(uv) = d(u)v + ud(v) &= u^p v + uv^p \\ &= u^p(v + u^{1-p}v^p) \\ &= u^p(\underbrace{v^{1-p} + u^{1-p}}_{1})v^p = u^p v^p. \end{aligned}$$

Thus, $d(uv) = u^p v^p = d(u)d(v)$.

Next,

$$\begin{aligned} d(u+v) &= (u+v)^p \\ &= \sum_{i=1}^{p} \begin{pmatrix} p \\ i \end{pmatrix} u^{p-i}v^i \\ &= \begin{pmatrix} p \\ 0 \end{pmatrix} u^p v^0 + \begin{pmatrix} p \\ 1 \end{pmatrix} u^{p-1}v + \cdots + \begin{pmatrix} p \\ p \end{pmatrix} u^0 v^p \\ &= u^p + \underbrace{pu^{p-1}v + \cdots}_{0} + v^p = u^p + v^p = d(u) + d(v). \end{aligned}$$

□

The next result presents a characterization of the direct products of classes of the near-rings constructed via matrix ring type near-rings.

Theorem 3.2.2. *Let* $\{\mathcal{N}_i\} = \{\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_h\}$ *be a family of classes of the near-rings constructed and define* $\mathcal{N}_1 = R_0 \oplus \mathcal{M}_1,\ \mathcal{N}_2 = R_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2,\ \cdots, \mathcal{N}_h = R_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_h$ *where* $\mathcal{M}_1 =< u_1 >$,
$\mathcal{M}_2 =< u_1, u_2 >, \cdots, \mathcal{M}_h =< u_1, \cdots, u_h >$ *and* $\mathcal{T} = \prod_{i=1}^{h} \mathcal{N}_i$ *be their direct products. Then, for any* $n \geq 1$, *the rings* $M_n(\mathcal{T})$ *and*
$S = \prod_{i=1}^{h} M_n(\mathcal{N}_i)$ *are isomorphic.*

Proof. Let $n \geq 1,\ \Theta = M_n(\mathcal{T}),\ \Omega = \prod_{i=1}^{h} M_n(\mathcal{N}_i)$. For any

$A = [a_{rs}]_{n\times n} \in \Theta$, let $a_{rs} = \prod_{i=1}^{h} a_{irs} \in \mathcal{T}$ for any $r, s \in \{1, \cdots, n\}$. Now, for each $i \in \{1, \cdots, h\}$, let $A_i = [a_{rs}]_{n\times n} \in M_n(\mathcal{N}_i)$, then, it is easy to verify that the map $f : \Theta \to \Omega$ with $f(A) = \prod_{i=1}^{h} A_i$ is an additive group isomorphism. To see that f is indeed a near-ring homomorphism, let $B = [b_{rs}]_{n\times n} \in \Theta$ and set $AB = C$, then $C = [c_{rs}]_{n\times n} \in \Theta$ where,

$$c_{rs} = \sum_{t=1}^{n} \Big(\prod_{i=1}^{h} a_{irt}\Big)\Big(\prod_{i=1}^{h} b_{irt}\Big) = \prod_{i=1}^{h} \Big(\sum_{t=1}^{n} a_{irt}b_{its}\Big) \in \mathcal{T}.$$

Hence, $c_{irs} = \sum_{t=1}^{n} a_{irt}b_{its}, \ : 1 \leq i \leq h, \ r, s \in \{1, \cdots, n\}$. Thus, by definition, $f(C) = \prod_i C_i$ where

$$C_i = [c_{irs}]_{n\times n} = [\sum_{t=1}^{n} a_{irt}b_{its}]_{n\times n} \in M_n(\mathcal{N}_i).$$

Dually,

$$f(A)f(B) = \prod_{i=1}^{h}(A_iB_i) = \prod_i \Big([a_{irs}][b_{its}]\Big) = \prod_i \Big[\sum_{t=1}^{n} a_{irs}b_{its}\Big] \in \Omega.$$

Thus, $f(AB) = f(A)f(B)$, as required. $\square$

3.3 Jordan Ideal and Commutation of $\mathcal{N}$ via the Frobenius Derivation

We investigate the commutation over the constructed $\mathcal{N}$ using the properties of the Jordan ideal, $J(\mathcal{N})$ and the Frobenius derivations d, d_1 and d_2 admitted by $\mathcal{N}$. In the sequel, the following results hold:

Theorem 3.3.1. *Let the Jordan ideal $J(\mathcal{N})$ of $\mathcal{N}$ be of the form:*
$J(\mathcal{N}) = (0) \oplus \sum_{i=1}^{h} \alpha_i u_i$ *and* $\Big(J(\mathcal{N})\Big)^2 = (0)$, *then the near-ring constructed has a 2-nilpotent radical.*

Proof. Suppose $u \in \mathcal{N}$ with $u = r_0 + \sum_{i=1}^{h} \alpha_i u_i$ and from the construction of $\mathcal{N}$, we have that $p^k u_i = 0$ for any prime p with $k = 1, 2$, so,

$2u = 2(r_0 + \sum_{i=1}^{h} \alpha_i u_i) = 0$, implies that $u = r_0 + \sum_{i=1}^{h} \alpha_i u_i \neq 0$, necessarily, thus $\mathcal{N}$ is zero-symmetric but non-2-torsion free. Now, let $d : \mathcal{N} \to \mathcal{N}$ be an identity Frobenius derivation obeying the product on $\mathcal{N}$, then the anti-commutator of u and itself is $u \circ u$ and given by:

$$\begin{aligned}
\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big) \circ \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big) &= \Big\{\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\} \\
&+ \Big\{\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\} \\
&= 2\Big\{\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\} \\
&= 2\underbrace{\Big(r_0^2 + \sum_{i=1}^{h} 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i \Big)}_{\in \mathcal{N}} = 0 \in J(\mathcal{N}).
\end{aligned}$$

Next, since the char $R_0 = char\mathcal{N} = p^k; k = 1, 2$, it is immediate that

$$\begin{aligned}
2\underbrace{\Big(r_0^2 + \sum_{i=1}^{h} 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i \Big)}_{\in \mathcal{N}} &= 2r_0^2 + 2\sum_{i=1}^{h} 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i \\
&= 0 \oplus \sum_{i=1}^{h} \alpha_i u_i \in J(\mathcal{N}).
\end{aligned}$$

So $J(\mathcal{N}) \cong (0) \oplus \sum_{i=1}^{h} \alpha_i u_i$.

Finally, since $2\Big\{\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\}$ is in $J(\mathcal{N})$,

$$\begin{aligned}
\Big(2\Big\{\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\}\Big)^2 &= \Big(2\underbrace{\Big(r_0^2 + \sum_{i=1}^{h} 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i \Big)}_{\in \mathcal{N}}\Big)^2 \\
&= 0.
\end{aligned}$$

Hence the required condition $\Big(J(\mathcal{N})\Big)^2 \cong (0)$. $\square$

Claim 3.3.1. *Let $\mathcal{N} = R_0 \oplus \mathcal{M}$ be the near-ring constructed in (3.2.1). Since $p^k u_i = \prod_{i=1}^{h} u_i = 0;\ k = 1, 2,$ it means that if $k = 2$, and $R_0 = \{0\}$ then $u^2 = 0$ so that $u\mathcal{N}u = 0$. So, $\mathcal{N}$ is said to be quasi-3 prime near ring of characteristic p or p^2.*

Next, we investigate some commutativity properties of $\mathcal{N}$.

Proposition 3.3.1. *Let $\mathcal{N}$ be the near-ring of the construction and $J(\mathcal{N})$ be its Jordan ideal. Then $J(\mathcal{N}) = \Big\{0 + \sum_{i=1}^{h} \alpha_i u_i\Big\} \subseteq C(\mathcal{N})$.*

Proof. By definition of center, we have that for all $\Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big) \in \mathcal{N}$

$$\begin{aligned} C(\mathcal{N}) &= \Big\{r_0 + \sum_{i=1}^{h} \alpha_i u_i : \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big)\Big\} \\ &= \Big\{\Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big)\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big\}. \end{aligned}$$

If $J(\mathcal{N}) = (0)$ we are done because, trivially $0 \in C(\mathcal{N})$ and thus $J(\mathcal{N}) \subseteq C(\mathcal{N})$. Otherwise, let $w \in J(\mathcal{N})$ and $v \in \mathcal{N}$, with $w = 0 + \sum_{i=1}^{h} \alpha_i u_i$ and $v = s_0 + \sum_{i=1}^{h} \beta_i u_i$, then

$$\begin{aligned} wv &= \Big(0 + \sum_{i=1}^{h} \alpha_i u_i\Big)\Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big) \\ &= 0 + \sum_{i=1}^{h} \Big\{\beta_i (0 + p^k R_0)^{d_i} + \alpha_i (s_0 + p^k R_0)^{d_i}\Big\} u_i \in J(\mathcal{N}), \end{aligned}$$

and

$$\begin{aligned} vw &= \Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big)\Big(0 + \sum_{i=1}^{h} \alpha_i u_i\Big) \\ &= 0 + \sum_{i=1}^{h} \Big\{\alpha_i (s_0 + p^k R_0)^{d_i} + \beta_i (0 + p^k R_0)^{d_i}\Big\} u_i \in J(\mathcal{N}). \end{aligned}$$

Since the usual addition is commutative on $\mathcal{N}$, we see that

$wv = vw \in J(\mathcal{N})$ and thus $w = \left(0 + \sum_{i=1}^{h} \alpha_i u_i\right) \in J(\mathcal{N})$ is also a member of $C(\mathcal{N})$, which clears the proof. Thus, $J(\mathcal{N}) \subseteq C(\mathcal{N})$. $\square$

Claim 3.3.2. *Given that $J(\mathcal{N}) \subseteq C(\mathcal{N})$, $\mathcal{N}$ is a commutative near-ring.*

Theorem 3.3.2. *Let $J(\mathcal{N}) \neq (0)$ be the Jordan ideal of the non-2-torsion free quasi 3-prime near-ring $\mathcal{N}$ constructed. Let $d : \mathcal{N} \to \mathcal{N}$ be the multiplicative Frobenius derivation of the construction 3.2.1, such that $d(J(\mathcal{N})) = 0$, then either $d = 0$ or $\left\{0 + \sum_{i=1}^{h} \alpha_i u_i\right\} \subseteq C(\mathcal{N})$.*

Proof. If $d = 0$, we are done. If $d \neq 0$, then the elements of $J(\mathcal{N})$ commute under the multiplication on $\mathcal{N}$. Therefore,

$$C(J(\mathcal{N})) \cap C(\mathcal{N}) = \left\{0 + \sum_{i=1}^{h} \alpha_i u_i\right\}.$$

So $(0 + \sum_{i=1}^{h} \alpha_i u_i) \in C(\mathcal{N})$ and $\left\{0 + \sum_{i=1}^{h} \alpha_i u_i\right\} \subseteq C(\mathcal{N})$. $\square$

Theorem 3.3.3. *Let $J(\mathcal{N}) \neq (0)$ be the Jordan ideal of the non-2-torsion free quasi 3-prime near-ring $\mathcal{N} = R_0 \oplus \mathcal{M}$. Let $d : \mathcal{N} \to \mathcal{N}$ be the multiplicative Frobenius derivation of the construction 3.2.1. If $u, v, w \in \mathcal{N}$ and $r_0, s_0, k_0 \in R_0$, $\left\{(r_0+pR_0)^d\gamma_i+(s_0+pR_0)^d\beta_i+(k_0+pR_0)^d\alpha_i\right\} \subseteq Ann(\mathcal{M})$, then $d(uvw) = d(r_0 s_0 k_0)$. Moreover,*

$$\Big(d(uv)\Big)w = \Big(d(u)v + ud(v)\Big)w = d(u)vw + ud(v)w.$$

Proof. Let $u, v, w \in \mathcal{N}$,

$$u = r_0 + \sum_{i=1}^{h} \alpha_i u_i, \; v = s_0 + \sum_{i=1}^{h} \beta_i u_i, \; w = k_0 + \sum_{i=1}^{h} \gamma_i u_i$$

so,

$$
\begin{aligned}
d(uvw) &= d\Big(r_0 s_0 k_0 + \sum_{i=1}^{h} \Big\{ \underbrace{(r_0 + pR_0)^d \gamma_i + (s_0 + pR_0)^d \beta_i + (k_0 + pR_0)^d \alpha_i}_{\sigma} \Big\} u_i \Big) \\
&= d\Big(r_0 s_0 k_0 + \sigma u_i\Big) = \Big(r_0 s_0 k_0 + \sigma u_i\Big)^p \\
&= \Big(r_0 s_0 k_0\Big)^p = d\Big(r_0 s_0 k_0\Big).
\end{aligned}
$$

Next, **case (i)**

$$
\begin{aligned}
d(uvw) &= d(uv)w + uvd(w) = \Big(d(u)v + ud(v)\Big)w + uvd(w) \\
&= \Big(u^p v + uv^p\Big)w + uvw^p = u^p vw + uv^p w + uvw^p.
\end{aligned}
$$

Also, **case (ii)**

$$
\begin{aligned}
d(uvw) &= d(u)vw + ud(vw) = d(u)vw + u\Big(d(v)w + vd(w)\Big) \\
&= d(u)vw + ud(v)w + uvd(w) = u^p vw + uv^p w + uvw^p
\end{aligned}
$$

From, (i) and (ii), it follows that

$$
\Big(d(uv)\Big)w = \Big(d(u)v + ud(v)\Big)w = d(u)vw + ud(v)w.
$$

□

Theorem 3.3.4. *Let $J(\mathcal{N})$ be the non-zero Jordan ideal of $\mathcal{N}$. Suppose $d_1 : \mathcal{N} \to \mathcal{N}$ and $d_2 : \mathcal{N} \to J(\mathcal{N})$ are two non-zero Frobenius derivations such that d_2 commutes on $J(\mathcal{N})$ and the commutators*
$[d_1(u), d_2(y)] = [u, y]$ where $u \in \mathcal{N}$ and $y \in J(\mathcal{N})$, then $d_1 \neq 0$ on $J(\mathcal{N})$ and $\mathcal{N}$ is commutative.

Proof. We show that $\mathcal{N}$ is indeed commutative iff $[u, y] = 0,\ \forall y \in J(\mathcal{N})$ whenever the Frobenius map $d_1 \neq 0$ on $J(\mathcal{N})$. Let $u \in \mathcal{N}$ and $y \in J(\mathcal{N})$. Since the ideal $J(\mathcal{N})$ is a subset of $\mathcal{N}$, $yu \in \mathcal{N}$ and $yu \in J(\mathcal{N})$. Now, $[d_1(u), d_2(y)] = [u, y]$ for all $u \in \mathcal{N}$ and $y \in J(\mathcal{N})$. Let $x = yu$ for some $x \in \mathcal{N}$. Then $[x, y] = [yu, y] = y[u, y]$.

Therefore $[d_1(yu), d_2(y)] = [yu, y] = y[u, y]$. But by definition of commutator, we see that

$$\begin{aligned} [d_1(yu), d_2(y)] &= d_1(yu)d_2(y) - d_2(y)d_1(yu) = y[d_1(u), d_2(y)] \\ &= y[u, y], \ \forall y \in J(N), u \in \mathcal{N}. \end{aligned}$$

From

$$[d_1(yu), d_2(y)] = d_1(yu)d_2(y) - d_2(y)d_1(yu), \tag{3.3.2}$$

we use d_1 and apply the definition of the multiplicative Frobenius derivation of the construction of $\mathcal{N}$ on the right hand side of equation (3.3.1) above to get,

$$[d_1(yu), d_2(y)] = \Big(yd_1(u) + d_1(y)u\Big)d_2(y) - d_2(y)\Big(yd_1(u) + d_1(y)u\Big) \tag{3.3.3}$$

$$= yd_1(u)d_2(y) + d_1(y)ud_2(y) - d_2(y)yd_1(u) - d_2(y)d_1(y)u$$

$$= yd_1(u)d_2(y) - yd_2(y)d_1(u),$$

because of the commuting property of d_2 on $J(\mathcal{N})$ and thus intuitively, equation (3.3.2) implies that,

$$d_1(y)ud_2(y) - d_2(y)d_1(y)u = \{0\} \Rightarrow d_1(y)ud_2(y) = d_2(y)d_1(y)u \tag{3.3.4}$$

Finally, consider some linear combination of u as $u = vm$ where
$u, v, m \in \mathcal{N}$ and by quasi-3-primeness of $\mathcal{N}$ together with the condition that

$d_1(y)ud_2(y) - d_2(y)d_1(y)u = \{0\}$, we see that,

$$d_1(y)vmd_2(y) - d_2(y)d_1(y)vm = \{0\} \Rightarrow d_1(y)\mathcal{N}[d_2(y), m] = \{0\} \forall y \in J(\mathcal{N}), m \in \mathcal{N}. \tag{3.3.5}$$

Thus $d_1(y) = 0$ or $d_2(y) \in C(\mathcal{N})$ by quasi-3 primeness of our $\mathcal{N}$.

If $d_1(y) \neq 0$ then $d_2(y) \in C(\mathcal{N})$ in which case $[u, y] = 0 \ \forall \ y \in J(\mathcal{N})$ which implies that $y \in C(\mathcal{N})$. We therefore conclude that $\mathcal{N}$ is commutative. □

The following results are useful in the next part:

Definition 3.3.1. *Given a zero symmetric near-ring $\mathcal{N}$, we say that $\mathcal{N}$ is integral if $xy = 0$ implies that $x = 0$ or $y = 0$, $xy \in \mathcal{N}$. We notice that $\mathcal{N} = R_0 \oplus U$ constructed above have zero divisors, thus non-integral and the zero divisors satisfy an ascending (reverse descending) chain conditions on their annihilators. The set*

$$\begin{aligned} \{x \in \mathcal{N} | fy \in \mathcal{N} \backslash 0 : yx = 0\} &= \{x \in \mathcal{N} | fy \in \mathcal{N} : xy = 0\} \\ &\Leftrightarrow Z_R(\mathcal{N}) \\ &= Z_L(\mathcal{N}) = J(\mathcal{N}) \end{aligned}$$

where $Z_R(\mathcal{N})$ are right zero divisors and $Z_L(\mathcal{N})$ are left zero divisors and $J(\mathcal{N})$ is the Jordan ideal.

Theorem 3.3.5. *Let $\mathcal{N}$ be the zero symmetric near-ring of the construction (3.2.1). Since $\mathcal{N}$ has the trivial and one maximal ideal, it is descending so that $\mathcal{N}Z_R = J(\mathcal{N})$ and if $m \notin Z_R(\mathcal{N})$, then $\mathcal{N}m = \mathcal{N}$.*

Proof. Forward inclusion $(\subseteq)$: Let $m \in \mathcal{N}^*$. Then any
$m^i \in \mathcal{N} \setminus Z_R \forall i \in \mathbb{N}$. The descending chain conditions on the ideals of $\mathcal{N}$ guarantee that $\mathcal{N}m \supseteq \mathcal{N}m^2 \supseteq ...$ terminates at some k-step say such that $\mathcal{N}m^k = \mathcal{N}m^{k+1}$: $k \in \mathbb{N}$ and thus

$$\mathcal{N}m^k = \mathcal{N}m^{k+1} = \mathcal{N}m(m^k).$$

This means that for any $x \in \mathcal{N}$ there exists some $i \in \mathcal{N}$ such that $im^k = (im)m^k$. Since $m^k \notin Z_R(\mathcal{N}) = J(\mathcal{N})$ we get that
$k = im \Rightarrow \mathcal{N} \subseteq \mathcal{N}m$. The reverse inclusion is immediate, that is
$\mathcal{N}m \subseteq \mathcal{N}$. Next, let $y \in Z_R(\mathcal{N})$ and $k \in \mathcal{N}$. If $k \in Z_R(\mathcal{N})$, then clearly $ky \in Z_R(\mathcal{N})$. If $k \in \mathcal{N}^*$, then $\mathcal{N}k = \mathcal{N}$ and since $y \in Z_R(N)$, there is an element $j \in \mathcal{N} \setminus \{0\}$ such that $jy = 0$. Thus $j = mk$ for some non-zero $m \in \mathcal{N}$ and consequently $jy = m(ky) = 0$..
$\therefore ky \in Z_r(\mathcal{N}) = J(\mathcal{N})$. From $m(ky) = 0 \Rightarrow \mathcal{N}Z_R(\mathcal{N}) = J(\mathcal{N})$ as required. $\square$

Theorem 3.3.6. *Let $\{\mathcal{N}\}$ be the set of near-rings of our construction and $z \in Z_R(\mathcal{N}) \setminus \{0\}$. Since every element of $Z_R(\mathcal{N})$ is nilpotent,*

$$J_2(\mathcal{N}) = Z_R(\mathcal{N})$$

where $J_2(\mathcal{N})$ is the Jacobson's radical of type 2 .

Proof. Suppose $\mathcal{N} = G\mathcal{N}(p^k, p^k) \cong GNF(p^{kr}), r = 1, k = 1$ then we know that $Z_r(\mathcal{N}_0) = \{0\}$ and each member of $Z_R(\mathcal{N}_0)$, which is the only member 0 has an index of nilpotence of 2. The statement is clear because $\mathcal{N}$ is a near-field. On the other hand, let $\mathcal{N} = \mathcal{R}_0 \oplus U$ be non-integral and let $z \in Z_R(\mathcal{N}) \setminus \{0\}$. Then there is a natural number k such that $z^k = 0$ but $z^{k-1} \neq 0$. Indeed $k \geq 2$ in this case. Then $z \in (0 : z)$ and $(0 : z) \leq Z_r(\mathcal{N})$. But $(0 : z)$ is a nil-ideal and therefore contained in $J_2(\mathcal{N})$. Thus $Z_R(\mathcal{N}) = J_2(\mathcal{N})$ $\square$

Remark 3.3.1. *Let $L = \{z \in \mathcal{N} \mid \mathcal{N}z \neq \mathcal{N}\}$. Then,*

(i). L is an $\mathcal{N}$- subgroup.

(ii). $\mathcal{N}$ is Local.

Proof. Let (ii)$\Rightarrow$ (i): If $\mathcal{N}$ is Local then $L = Z_R(\mathcal{N})$ by definition : If $z \in Z_R(\mathcal{N})$

then $z^n \in Z_R(\mathcal{N}) : n \in \mathbb{N}n \geq 2$. Therefore z must be nilpotent, otherwise, the semi-group generated by z contains a non-zero idempotent. $\square$

Let $\mathcal{N}^*$ be the set of all the invertible elements with respect to near-ring multiplication given, so $\mathcal{N}^* = \mathcal{N} \setminus Z_R(\mathcal{N})$. We want to find a lower bound for $\mid \mathcal{N}^* \mid = \mid < \alpha > \times 1 + J(\mathcal{N}) \mid$ where $\mid < \alpha > \mid = \varphi(p^r - 1)$ as a consequence obtaining an upper bound for $\mid Z_R(\mathcal{N}) \mid$.

3.4 Some Graph Morphisms of $\Gamma(\mathcal{N})$

In this section, we determine some aspects of graph morphisms over $\mathcal{N}$ in construction (3.2.1).

Proposition 3.4.1. *Let $\mathcal{N}$ be a local near ring of characteristic p, defined with respect to multiplication in construction (3.2.1). Then,*

$$|Aut(\Gamma(\mathcal{N}))| = \begin{cases} (p^{hr} - 1)! \; or. \\ (p^{hr} - 2)! \sum_{i=1}^{hr} \varphi(p^i) \; or \\ \frac{1}{p^{hr-1}} \varphi(p^{hr})(p^{hr} - 2)! \sum_{i=1}^{hr} p^{hr-i}. \end{cases}$$

Proof. From the construction, we have that $R_0 = G\mathcal{N}(p^r, p)$.
Let $K = R_0/pR_0$ be a near-field. Suppose $U = K^h$ is an R_0-module generated by $\{u_1, u_2, u_3, \cdots, u_h\}$. Therefore $\mathcal{N} = R_0 \oplus U$ is an additive group. But $Z_L(\mathcal{N}) = R_0u_1 \oplus R_0u_2 \oplus \cdots \oplus R_0u_h$ and every element in $Z_L(\mathcal{N})^*$ is of the form $(0, a_1, a_2, \cdots, a_h)$ so that the product of every pair $(0, a_1, a_2, \cdots, a_h), (0, b_1, b_2, \cdots, b_h) \in Z_L(\mathcal{N})^*$ is identically zero, indicating that every pair of elements of zero divisor graph of $\mathcal{N}$ are joined by an edge. So $|V(\Gamma(\mathcal{N}))| = |Z(\mathcal{N})^*| = p^{hr} - 1$ so that $Aut(\Gamma(\mathcal{N})) \cong S_{p^{hr}-1}$ and the first part of the results follows.
Next,

$|Aut(\Gamma(\mathcal{N}))| = (p^{hr}-1)! = (p^{hr}-1)(p^{hr}-2)!$ and

$$\begin{aligned}
\sum_{i=1}^{hr} \varphi(p^i) &= \varphi(p^1)+\varphi(p^2)+\varphi(p^3)+\cdots+\varphi(p^{hr}) \\
&= (p-1)+p(p-1)+p^2(p-1)+p^3(p-1)+\cdots+p^{hr-1}(p-1) \\
&= (p-1)(1+p+p^2+p^3+\cdots+p^{hr-1}) \\
&= (p-1)(\frac{p^{hr}-1}{p-1}) = p^{hr}-1.
\end{aligned}$$

Dividing $|Aut(\Gamma(\mathcal{N})|$ by $\sum_{i=1}^{hr}\varphi(p^i)$ gives the relation

$$|Aut(\Gamma(\mathcal{N}))| = (p^{hr}-2)!\sum_{i=1}^{hr}\varphi(p^i).$$

Finally,

$$\begin{aligned}
|Aut(\Gamma(\mathcal{N}))| &= (p^{hr}-1)! = (p^{hr}-1)(p^{hr}-2)! \\
&= (p-1)\sum_{i=1}^{hr} p^{hr-i}(p^{hr}-2)!
\end{aligned}$$

and $\varphi(p^{hr}) = p^{hr}-p^{hr-1} = p^{hr-1}(p-1)$. On dividing $|Aut(\Gamma(\mathcal{N}))|$ by $\varphi(p^{hr})$ and expressing the equation in terms of $|Aut(\Gamma(\mathcal{N}))|$ establishes the relation

$$|Aut(\Gamma(\mathcal{N}))| = \frac{1}{p^{hr-1}}\varphi(p^{hr})(p^{hr}-2)!\sum_{i=1}^{hr} p^{hr-i}.$$

$\square$

Proposition 3.4.2. *Let $\mathcal{N}$ be a local near ring of characteristic p, with respect to the multiplication in construction (3.2.1). Then,*

$$|Aut(\Gamma(\mathcal{N}))| = \begin{cases} (p^{hr}-3)!\sum_{v\in V} deg(v). \\ 2(p^{hr}-3)!|E|. \end{cases}$$

Proof. Note that, $|Aut(\Gamma(\mathcal{N}))| = (p^{hr}-1)!$ is well known and the sum of degrees of zero divisor graph of $\mathcal{N}$ is $(p^{hr}-1)(p^{hr}-2)$ accordingly. Since $\frac{(p^{hr}-1)!}{(p^{hr}-1)(p^{hr}-2)} = (p^{hr}-3)!$, it follows that

$$|Aut(\Gamma(\mathcal{N}))| = (p^{hr}-3)! \sum_{v\in V} deg(v).$$

Also, $|Aut(\Gamma(\mathcal{N}))| = (p^{hr}-1)!$ and as such, the sum of edges of $\Gamma(\mathcal{N})$ is $\frac{1}{2}(p^{hr}-1)(p^{hr}-2)$. Since $\frac{(p^{hr}-1)!}{\frac{1}{2}(p^{hr}-1)(p^{hr}-2)} = 2(p^{hr}-3)!$, straight forward argument gives;

$$|Aut(\Gamma(\mathcal{N}))| = 2(p^{hr}-3)!|E|.$$

□

Proposition 3.4.3. *Let $\mathcal{N}$ be a class of near ring constructed, of characteristic p such that $p \in \mathcal{J}(\mathcal{N})$. Then,* $|V(\Gamma(\mathcal{N}))| = \begin{cases} \frac{1}{p^{hr}-2}\sum_{v\in V} deg(v). \\ \frac{2|E|}{p^{hr}-2}. \end{cases}$

Proof. By definition, $|Z(\mathcal{N})^*| = |V(\Gamma(\mathcal{N}))| = p^{hr}-1$. The sum of the degrees of $\Gamma(\mathcal{N})$ is $(p^{hr}-1)(p^{hr}-2)$. Since $\frac{(p^{hr}-1)}{(p^{hr}-1)(p^{hr}-2)} = \frac{1}{(p^{hr}-2)}$ the first part follows.

Next, the fact that the sum of edges of $\Gamma(\mathcal{N})$ is $\frac{1}{2}(p^{hr}-1)(p^{hr}-2)$ follows from the previous proposition while

$$|Z(\mathcal{N})^*| = |V(\Gamma(\mathcal{N}))| = p^{hr}-1$$

is clear. Now, writing $\frac{(p^{hr}-1)}{\frac{1}{2}(p^{hr}-1)(p^{hr}-2)} = \frac{2}{(p^{hr}-2)}$ clears the proof. □

Proposition 3.4.4. *Let $\mathcal{N}$ be near ring constructed of characteristic p^2. Then,*

$$|Aut(\Gamma(\mathcal{N}))| = \begin{cases} (p^{(h+1)r}-1)! \\ (p^{(h+1)r}-2)!\sum_{i=1}^{(h+1)r}\varphi(p^i) \\ \frac{1}{p^{(h+1)r-1}}\varphi(p^{(h+1)r})(p^{(h+1)r}-2)!\sum_{i=1}^{(h+1)r} p^{(h+1)r-i}. \end{cases}$$

Proof. Define the Galois subnear-ring by $R_0 = GR(p^{2r}, p^2)$ and $K = R_0/pR_0$. Let $U = K^h$ be an R_0-module generated by $\{u_1, u_2, u_3, \cdots, u_h\}$ and $\mathcal{N} = R_0 \oplus U$ is an additive group.

Clearly, $Z_L(\mathcal{N}) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus \cdots \oplus R_0u_h$ and the product of every pair of elements in $Z_L(\mathcal{N})$ is zero. Thus $|\mathcal{N}| = |R_0||U| = p^{(h+2)r} \Rightarrow |Z_L(\mathcal{N})| = p^{(h+1)r}$ and $|V(\Gamma(\mathcal{N}))| = |Z_L(\mathcal{N})^*| = p^{(h+1)r} - 1$. Since every vertex of $(Z_L\mathcal{N})^*$ is adjacent to all the other vertices of the zero divisor graph of $\mathcal{N}$, so $Aut(\Gamma(\mathcal{N}))$ must permute all the symmetries of $\Gamma(\mathcal{N})$ independently so that $Aut(\Gamma(\mathcal{N})) \cong S_{p^{(h+1)r}-1}$.

Next, $|Aut(\Gamma(\mathcal{N}))| = (p^{(h+1)r}-1)! = (p^{(h+1)r}-1)(p^{(h+1)r}-2)!$ and $\sum_{i=1}^{(h+1)r} \varphi(p^i) = p^{(h+1)r} - 1$, so that $|Aut(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 2)! \sum_{i=1}^{(h+1)r} \varphi(p^i)$.

Finally,

$$\begin{aligned} |Aut(\Gamma(\mathcal{N}))| &= (p^{(h+1)r} - 1)! \\ &= (p^{(h+1)r} - 1)(p^{(h+1)r} - 2)! \\ &= (p-1) \sum_{i=1}^{(h+1)r} p^{(h+1)r-i}(p^{(h+1)r} - 2)! \end{aligned}$$

and $\varphi(p^{(h+1)r}) = p^{(h+1)r} - p^{(h+1)r-1} = p^{(h+1)r-1}(p-1)$.

Giving the relation in terms of $|Aut(\Gamma(\mathcal{N})|$ yields,

$$|Aut(\Gamma(\mathcal{N}))| = \frac{1}{p^{(h+1)r-1}} \varphi(p^{(h+1)r})(p^{(h+1)r} - 2)! \sum_{i=1}^{(h+1)r} p^{(h+1)r-i}.$$

□

Proposition 3.4.5. *Let $\mathcal{N}$ be a near ring of characteristic p^2, with respect to multiplication in construction (3.2.1). Then*

$$|V(\Gamma(\mathcal{N}))| = \begin{cases} \frac{1}{p^{(h+1)r-1}} \varphi(p^{(h+1)r}) \sum_{i=1}^{(h+1)r} p^{(h+1)r-i}. \\ \sum_{i=1}^{(h+1)r} \varphi(p^i). \end{cases}$$

Proof.

$$|V(\Gamma(\mathcal{N}))| = p^{(h+1)r} - 1 = (p-1)(p^{(h+1)r-1} + p^{(h+1)r-2} + p^{(h+1)r-3} + \cdots + 1)$$

$$= (p-1)\sum_{i=1}^{(h+1)r} p^{(h+1)r-i}.$$

Also,

$$\varphi(p^{(h+1)r}) = p^{(h+1)r} - p^{(h+1)r-1} = p^{(h+1)r-1}(p-1),$$

so that

$$|V(\Gamma(\mathcal{N}))| = \frac{1}{p^{(h+1)r-1}}\varphi(p^{(h+1)r})\sum_{i=1}^{(h+1)r} p^{(h+1)r-i}.$$

Next,

$$|V(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 1)$$

and $\sum_{i=1}^{(h+1)r}\varphi(p^i) = p^{(h+1)r} - 1$. Thus,

$$|V(\Gamma(\mathcal{N}))| = \sum_{i=1}^{(h+1)r}\varphi(p^i).$$

□

Lemma 3.4.1. *Let $\mathcal{N}$ be a near ring of characteristic p^2, with respect to the multiplication in construction (3.2.1). Then,$|E| = \frac{1}{2}(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)$. Moreover,*

$$\sum_{v\in V} deg(v) \quad = \quad (p^{(h+1)r} - 1)(p^{(h+1)r} - 2).$$

The lemma above will be used in the proof of the next result.

Proposition 3.4.6. *Let $\mathcal{N}$ be the near ring constructed, then,*

$$|Aut(\Gamma(\mathcal{N}))| = \begin{cases} (p^{(h+1)r} - 3)!\sum_{v\in V} deg(v). \\ 2(p^{(h+1)r} - 3)!|E|. \end{cases}$$

Proof. Evidently,

$$|Aut(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 1)!$$

and

$$|Z(\mathcal{N})^*| = |V(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 1).$$

The sum of degrees of $\Gamma(\mathcal{N})$ is $(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)$.
Since $\frac{(p^{(h+1)r}-1)!}{(p^{(h+1)r}-1)(p^{(h+1)r}-2)} = (p^{(h+1)r} - 3)!$, it follows that $|Aut(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 3)! \sum_{v\in V} deg(v)$.

Also, $|Aut(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 1)!$
and $|Z(\mathcal{N})^*| = |V(\Gamma(\mathcal{N}))| = (p^{(h+1)r} - 1)$. But the sum of the edges of $\Gamma(\mathcal{N})$ is $\frac{1}{2}(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)$.
Since $\frac{(p^{(h+1)r}-1)!}{\frac{1}{2}(p^{(h+1)r}-1)(p^{(h+1)r}-2)} = 2(p^{(h+1)r} - 3)!$ we have $|Aut(\Gamma(\mathcal{N}))| = 2(p^{(h+1)r} - 3)!|E|$. □

Proposition 3.4.7. *Let $\mathcal{N}$ be a a near ring of characteristic p^2, with respect to the multiplication in construction (3.2.1) . Then,*

$$|V(\Gamma(\mathcal{N}))| = \begin{cases} \frac{1}{p^{(h+)r}-2} \sum_{v\in V} deg(v). \\ \frac{2|E|}{p^{(h+1)r}-2}. \end{cases}$$

Proof. Let $|Z(\mathcal{N})^*| = |V(\Gamma(\mathcal{N})| = p^{(h+1)r} - 1$ and the sum of the degrees of $\Gamma(\mathcal{N})$ is $(p^{(h+1)r} - 1)(p^{(h+1)r} - 2)$. Since $\frac{(p^{(h+1)r}-1)}{(p^{(h+1)r}-1)(p^{(h+1)r}-2)} = \frac{1}{(p^{(h+1)r}-2)}$ we have $\frac{1}{p^{(h+)r}-2} \sum_{v\in V} deg(v)$.

Next, $|Z(\mathcal{N})^*| = |V(\Gamma(\mathcal{N})| = p^{(h+1)r} - 1$ and the sum of edges of $\Gamma(\mathcal{N})$ is $\frac{1}{2}(p^{(h+1)r}-1)(p^{(h+1)r}-2)$. Since $\frac{(p^{(h+1)r}-1)}{\frac{1}{2}(p^{(h+1)r}-1)(p^{(h+1)r}-2)} = \frac{2}{(p^{(h+1)r}-2)}$ we get $|V(\Gamma(\mathcal{N}))| = \frac{2}{p^{(h+1)r}-2}|E|$. □

CHAPTER FOUR

ZERO SYMMETRIC LOCAL NEAR-RINGS OF CONSTRUCTION II

4.1 Introduction

This chapter contains the construction of zero symmetric local near-rings of $charp^k$: $k \geq 3$ using the elements of the sub near-modules obtained from Galois near-rings. Furthermore, the properties of regular elements of Von Neumann are investigated.

4.2 The Construction

Let $R_0 = G\mathcal{N}(p^{kr}, p^k)$. Let $i = 1, ..., h$ and $u_i \in Z_L(\mathcal{N})$ and $\mathcal{M} =< u_i >$.
Then,

$$\mathcal{N} = R_0 \oplus \mathcal{M} = R_0 \oplus \sum_{i=1}^{h} (R_0/pR_0)^i$$

is a group with respect to addition.
On $\mathcal{N}$, let

$$(r_0, \overline{r}_1, ..., \overline{r}_h)(s_0, \overline{s}_1, ..., \overline{s}_h) = (r_0 s_0, r_0\overline{s}_1 + \overline{r}_1 s_0, ..., r_0\overline{s}_h + \overline{r}_h s_0)^{\delta} \tag{4.2.1}$$

where δ is the identity Frobenius automorphism. The multiplication turns $\mathcal{N}$ into a local zero symmetric near-ring with identity $(1, \overline{0}, ..., \overline{0})$.

Remark 4.2.1. *By remark 4.2 of construction (4.2.1), $\mathcal{N} = R_0 \oplus \mathcal{M}$ is commutative since δ is the identity Frobenius automorphism.*

Proposition 4.2.1. *Consider $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k)$ where $k \geq 3$. Then, $char\mathcal{N} = p^k$*

and:

(i). $Z_L(\mathcal{N}) = pR_0 \oplus \sum_{i=1}^{h}(R_o/pR_o)^i$

(ii). $(Z_L(\mathcal{N}))^{k-1} = p^{k-1}R_0 \neq (0)$

(iii). $(Z_L(\mathcal{N}))^k = (0)$.

Proof. Char $G\mathcal{N}(p^{kr}, p^k) = char\mathcal{N}$ and $id_{\mathcal{N}} = id_{G\mathcal{N}(p^{kr}, p^k)}$.
Let $a \in R_0$ and a not contained in pR_o and let $s \in Z_L(\mathcal{N})$.
Then

$$\begin{aligned}(a+s)^{pr} &= a^{pr} + s' : (s' \in Z_L(\mathcal{N})) \\ &= (a+s'')^{p^r-1} : (s'' \in Z_L(\mathcal{N})).\end{aligned}$$

But $(a+s'')^{p^r-1} \equiv 1 + s'''$ with $s''' \in Z_L(\mathcal{N})$ and $(1+s''')^{p^k-1} = 1$. Hence $(a+s)$ is regular and not zero.
Since $\mid Z_L(\mathcal{N}) \mid= p^{(h+k-1)r}$ and
$\mid (R_0/pR_0)^* + Z_L(\mathcal{N}) \mid= (p^r-1)(p^{(h+k-1)r})$, it follows that
$(R_0/pR_0)^* + Z_L(\mathcal{N}) = \mathcal{N} - Z_L(\mathcal{N})$ and hence all the elements outside $Z_L(\mathcal{N}) \setminus \{0\}$ are regular.

$\square$

4.3 Regular Elements Determined by Von Neumann Inverses

This section characterizes the properties of regular elements, structures and orders as well as the counting of Von Neumann and Reflexive inverses of the near rings of constructions (3.2.1) and (4.2.1).

In the zero-symmetric near-rings considered in this thesis, Von-Neumann regular

elements are determined by their inner inverses. Let $R(\mathcal{N})$ be the set of regular elements of $\mathcal{N}$, for elements
$x, y \in R(\mathcal{N}),\ x = y \Leftrightarrow I(x) = I(y)$ where I denotes the set of inner inverses.

Remark 4.3.1. *A regular element $x \in R(\mathcal{N})$ may have more than one Von-Neumann inverse. However, for the classes of near-rings considered in this study, the Von-Neumann inverses are unique.*

Definition 4.3.1. *An element $y \in \mathcal{N}$ is an outer inverse of $x \in R(\mathcal{N})$ if and only if $yxy = y$ and whenever $y \in I(x)$, then yxy is both inner and outer. Indeed $y \in \mathcal{N}$ is called reflexive inverse of $x \in R(\mathcal{N})$ if it is both inner and outer.*

Let $Ref(a)$ be the set of all reflexive inverses of $x \in R(\mathcal{N})$, $I(x)$ be the set of inner inverse of $x \in R(\mathcal{N})$, then, the following results hold:

Proposition 4.3.1. *Let $\mathcal{N}$ be a class of near-ring of the construction (4.2.1). For $x \in \mathcal{N}$ and $x_0 \in I(x)$, we have that:*

$$I(x) = \{x_0 + \alpha - x_0 x \alpha x x_0 \mid \alpha \in \mathcal{N}\}.$$

Proof. From the construction, if $x \in \mathcal{N}$, then

$$x = (r_0 + (\sum_{i=1}^{h} r_0 + pr^{'})r^{'} \in G\mathcal{N}(p^{kr}, p^k)/pG\mathcal{N}(p^{kr}, p^k)).$$

So the definition of the multiplication in $\mathcal{N}$ gives the desired result. □

Denote by $l(x)$ and $r(x)$ the left and the right annihilator of an element $x \in \mathcal{N}$. So the inner annihilator of $x \in \mathcal{N}$ is:
$Iann(x) = \{y \in \mathcal{N} : xyx = 0\}$.

Theorem 4.3.1. *Let $\mathcal{N}$ be the near ring of construction (4.2.1). If $a \in R(\mathcal{N})$, then for any $b \in \mathcal{N}$, $bI(a)b$ is a singleton set if and only if $b \in \mathcal{N}a \cap a\mathcal{N}$.*

Proof. Suppose there exists $x, y \in \mathcal{N}$ such that $b = xa = ay$ and let $a_0 \in I(a)$. We then have that for any $t \in \mathcal{N}$,

$$\begin{aligned} b(a_0 + t - a_0ataa_o)b &= (xaa_0 + xat - xataa_0)ay \\ &= xay + xatay - xatay \\ &= xay. \end{aligned}$$

Thus the set $bI(a)b = \{xay\}$ is singleton.

Conversely, suppose that $bI(a)b = \{ba_0b\}$.

We then have: $b(a_0 + t - a_oataa_0)b = ba_0b$ for any $t \in \mathcal{N}$. This implies that for any $t \in \mathcal{N}$, we have:

$$b(t - a_0ataa_0)b = 0 \tag{4.3.2}$$

Substituting $(1 - a_0a)t$ for t in this equality yields $b(1 - a_0ataa_0)tb = 0$ for any $t \in \mathcal{N}$. But $\mathcal{N}$ constructed is semiprime so that

$$b(1 - a_0a) = 0 \Rightarrow b = ba_0a \in \mathcal{N}a \tag{4.3.3}$$

Similarly, substituting t by $t(1 - aa_0)$ in the equality (i)

gives

$$b = aa_0b \in a\mathcal{N} \tag{4.3.4}$$

Comparing (4.3.2) and (4.3.3), we conclude that $b \in \mathcal{N}a \cap a\mathcal{N}$ □

Lemma 4.3.1. *Let $\mathcal{N}$ be the near ring constructed and let $b, d \in \mathcal{N}$ such that $b + d$ is a Von Neumann regular element. Then the following are equivalent:*

(i) $b\mathcal{N} \oplus d\mathcal{N} = (b + d)\mathcal{N}$

(ii) $\mathcal{N}b \oplus \mathcal{N}d = \mathcal{N}(b + d)$

(iii) $b\mathcal{N}b \cap d\mathcal{N} = \{0\}$ *and* $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

The next result shows when $I(a) \subseteq I(b)$ necessarily and sufficiently where $a, b \in \mathcal{N}$

Proposition 4.3.2. *Let* $a, b \in R(\mathcal{N})$. *Then* $I(a) \subseteq I(b)$ *if and only if* $b\mathcal{N} \cap d\mathcal{N} = \{0\}$ *and* $\mathcal{N}b \cap \mathcal{N}d = \{0\}$ *where* $a = d + b$.

Proof. Let $I(a) \subseteq I(b)$. Then by definition, there exists some $x \in I(a)$ such that $bxb = b$.
Now $b \in \mathcal{N}a \cap a\mathcal{N}$.
Write $b = \alpha a = a\beta$ where $\alpha, \beta \in \mathcal{N}$.
Then $bI(a)a = b$.
Next

$$\begin{aligned} bI(a)d &= bI(a)a - bI(a)b \\ &= b - bI(a)b = 0. \end{aligned}$$

Consider now

$$\begin{aligned} dI(a)b &= aI(a)b - bI(a)b \\ &= \alpha\beta - bI(a)b \\ &= b - b = 0. \end{aligned}$$

We thus have $bI(a)d = 0$ and $dI(a)b = 0$ Then for any $x \in I(a)$ we have;

$$\begin{aligned} b + d = a &= axa \\ &= (b+d)x(b+d) \\ &= bxa + dxb + dxd \\ &= b + 0 + dxd. \end{aligned}$$

This yields

$$dI(a)d = d \tag{4.3.5}$$

To show that $d\mathcal{N} \cap b\mathcal{N} = \{0\}$.

Let $bx = dy \in b\mathcal{N} \cap d\mathcal{N}$.

Multiplying both sides of (4.3.4) by y on the right and using $bx = dy$ yields, $dI(a)bx = dy$.

But from above we have that $dI(a)b = 0$ and so $dy = 0$.

Similarly, we show that $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

Let $xb = yd \in \mathcal{N}b \cap \mathcal{N}d$. Multiplying both sides of (4.3.4) on the left by y. We get: $ydI(a)d = yd$. This proves that $xbI(a)d = yd$.

Since $bI(a)d = 0$, we obtain $yd = 0$, showing that $\mathcal{N}b \cap \mathcal{N}d = \{0\}$.

□

Theorem 4.3.2. *Let $a, b \in R(\mathcal{N})$. Then $I(a) = I(b)$ if and only if $a = b$.*

Proof. From the construction, $\mathcal{N} = Z_L(\mathcal{N}) \cup \underbrace{\mathcal{N}^* \cup \{0\}}$. We can write $a = b + d$ with $b\mathcal{N} \cap d\mathcal{N} = 0$ and $\mathcal{N}d \cap \mathcal{N}d = 0$. But $(b+d)\mathcal{N} = b\mathcal{N} \oplus d\mathcal{N}$. Since $I(a) = I(b)$, we have that $aI(b)a = \{a\}$ and $bI(a)b = \{b\}$ and therefore it follows that $\mathcal{N}a = \mathcal{N}b$ and $a\mathcal{N} = b\mathcal{N}$ which leads to $a\mathcal{N} = (b + d)\mathcal{N} = b\mathcal{N} \oplus d\mathcal{N}$, giving $d = 0$. Hence $a = b$ as desired.

□

Next, we provide the analogue to the previous theorem by generalizing the case to reflexive inverses:

Theorem 4.3.3. *Let $a, b \in R(\mathcal{N})$. Then $Ref(a) = Ref(b)$ if and only if $a = b$.*

Proof. Let $a_o \in Ref(a) = Ref(b)$. Since $a = 0$ if and only if $Ref(a) = 0$, assume that $a, b \neq 0$.

Since $bRef(a)b = bRe(b)b = b$ and $Ref(a) = I(a)aI(a)$, we have that for any

$t \in \mathcal{N}$, $b(a_0 + t - a_0ataa_o)a(a_0 + t - a_oataa_0)b = b$. Replacing t by $(1 - a_0a)t$ and noting that $a(1 - a_0a) = 0$,

we obtain successively

$b(a_0a + (1 - a_0a)ta)(a_0 + (1 - a_0a)t)b = b$ and

$b(a_0b + (1 - a_0a)ta)(a_0)b = b$ and so $ba_0b + b(1 - a_0a)taa_0)b = b$.

Since $ba_0b = b$ gives $b(1 - a_0a)taa_0b = 0 \ \forall \ t \in \mathcal{N}$, this leads to

$aa_0b(1 - a_0a)taa_0b(1 - a_0a) = 0 \ \forall \ t \in \mathcal{N}$.

But we are guaranteed of semi-primeness of $\mathcal{N}$ which then implies that

$aa_0b(1 - a_0a) = 0$. Left multiplying by $a_0 \in Ref(a)$, we get that

$a_0b(1 - a_0a) = 0$ and hence since $a_0 \in I(b)$, we conclude that $b(1 - a_0a) = 0$.

Therefore we obtain that $\mathcal{N}b \subseteq \mathcal{N}a$ and $\mathcal{N}a \subseteq \mathcal{N}b$ which implies that $\mathcal{N}a = \mathcal{N}b$.

Finally, Hartwig's theorem gives us that there exist invertible elements $u, v \in \mathcal{N}$ such that $a = bu$ and $b = av$ $\square$

4.4 Structures and Orders of Von-Neumann Regular Elements

Definition 4.4.1. *Let $(\mathcal{N}, +)$ be a group. The exponent of the group is the least common multiple of all the orders of the group elements.*

Remark 4.4.1. *Let $\mathcal{N}$ be a finite near-ring with identity 1 and n be the exponent of $(\mathcal{N}, +)$. Then $ord(1) = n$.*

Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $\mid \mathbb{Z}_n^* \mid= \varphi(n)$, φ- being the Euler-Phi function. We now give a generalization of this result to an arbitrary case:

Proposition 4.4.1. *Let $\mathcal{N}$ be the near-ring from the classes of near-rings in construction (3.2.1) and (4.2.1) and $\mathcal{N}^*$ be as obtained in the constructions. Let n be the exponent of $(\mathcal{N}, +)$ and φ be the Euler's-Phi function. Then there is a subgroup of order $\varphi(n)$ contained in $\mathcal{N}^*$.*

Proof. We use the fact that the identity $(1,0,0,...,0) \in \mathcal{N}$ generates a subring of $\mathcal{N}$. Assume the usual (+) and the multiplication (.) defined on $\mathcal{N}$. Consider the cyclic group $< 1,0,0,...,0 >$, additively generated by 1 where $1 \equiv (1,0,0,...,0)$. Then $l.1 = \underbrace{1+1+...+1}\, l - summands$ and $k.1 = \underbrace{1+1+...+1}\, k - summands$ are two elements of $< 1 >$. Since 1 is an identity: $(l.1)(k.1) = (lk.1) \in < 1 >$. Thus $S = (< 1 >, +, .)$ is a sub-near ring containing the identity. Indeed $f : S \longrightarrow \mathbb{Z}_n : f(k.1) = [k]_n$ is a near-ring isomorphism. Thus $S \cong \mathbb{Z}_n$. Let S^* be the group of units of S. It follows from the canonical isomorphism above that S^* has $\varphi(n)$ invertible elements. Since S and $\mathcal{N}$ have the same identity elements, an element $y \in S : y^{-1} \in S$ implies that $y^{-1} \in \mathcal{N}$.

$\therefore S^* \subseteq \mathcal{N}^*$ and S^* is a subgroup of order $\varphi(n)$. □

Corollary 4.4.1. $\mid R(\mathcal{N}) \mid = \varphi(n) + 1$.

We recall some notions in Number Theory: Let $\mathcal{N} = \mathbb{Z}_{p^k}$. For each natural number n, we have the following functions

$\varphi(n) = \{\sharp x : 1 \leq x \leq n \ \gcd(x,n) = 1\}$ $\overline{w}(n)$ = number of distinct primes dividing n

$\tau(n)$ = number of the divisors of n and

$\sigma(n)$ = sum of the divisors of n

Let $p = 2$ and $k = 2$ so that $n = 4 \Rightarrow n = p^k$

Then $\varphi(4) = 2$, $\overline{w}(4) = 1$, $\tau(4) = 3$ and

$\sigma(4) = 1 + 2 + 4 = 7$.

Theorem 4.4.1. *([55], Theorem 2) Let p be a prime integer and $k \in \mathbb{Z}^+$ then $a \in G\mathcal{N}(p^k, p^k)$ is regular if $a^{p^k - p^{k-1} + 1} \cong a(mod\ p^k)$.*

The element $a^{p^k - p^{k-1} + 1}$ is a Von Neumann inverse of a

Example 4.4.1. *Let $\mathcal{N} = \mathbb{Z}_4[x] / < x + 1 >$. Then $\mathcal{N} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$.*

From $a \in R(\mathcal{N})$ if and only if $a^{p^k - p^{k-1} + 1} \equiv a(mod\ p^k)$ gives:

If $a = \bar{3}$*, then,* $\bar{3}^{2^2-2^{2-1}+1} \equiv \bar{3}(mod\ 4)$ *which implies that* $(\bar{3})^3 \equiv \bar{3}(mod\ 4)$*.*

Thus $\bar{3}$ *is a regular element and* $(\bar{3})^3$ *is a Von-Neumann inverse. Therefore, Von-Neumann inverses of* $\bar{1}$, $\bar{3}$ *are* $\bar{1}$, $\bar{3}$ *respectively.*

Theorem 4.4.2. *Let* $\mathcal{N} = G\mathcal{N}(p^k, p^k)$*. Then,*

$$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$$

Proof. Since $\mathcal{N} = G\mathcal{N}(p^k, p^k)$ is zero-symmetric local, every element
$a \in R(\mathcal{N})$ is either 0 or a unit.
But $\mid \mathcal{N}^* \mid = p^k - p^{k-1}$ and the zero element is unique, it follows from the arithmetic function formula that:

$$V(p^k) = p^k - p^{k-1} + 1 = \varphi(p^k) + 1.$$

□

Definition 4.4.2. *Let* $x, y \in \mathbb{Z}^+$*. We say that* x *is a unitary divisor of* y *if* $x \mid y$ *and* $\gcd(x, \frac{y}{x}) = 1$ *and we write* $x \| y$*.*

The number of regular elements in $\mathcal{N}$ can then be calculated using the unitary divisors of an integer $n = \mid \mathcal{N} \mid$.

Proposition 4.4.2. *Let* $\mathcal{N} = G\mathcal{N}(p^k, p^k)$*. Then* $V(\mathcal{N}) = \Sigma_{x\|p^k}\varphi(x)$ *and* $V(N)/\varphi(p^k) = \Sigma_{x\|p^k}\frac{1}{\varphi(x)}$*.*

Proof. In $\mathcal{N}$ above $x = 1$ and $x = p^k \equiv 0(mod\ p^k)$.
By definition, $\varphi(1) = 1$. But $\varphi(p^k) = p^k - p^{k-1}$ and

$$\begin{aligned} V(p^k) &= p^k - p^{k-1} + 1 \\ &= \varphi(p^k) + \varphi(1). \end{aligned}$$

Moreover,

$$\begin{aligned}\frac{V(p^k)}{\varphi(p^k)} &= \frac{p^k - p^{k-1} + 1}{p^k - p^{k-1}} \\ &= 1 + \frac{1}{p^k - p^{k-1}} \\ &= \frac{1}{\varphi(1)} + \frac{1}{\varphi(p^k)}.\end{aligned}$$

The summatory function:

$$\begin{aligned}K(p^k) &= \sum_{x\|(p^k)} V(x) \\ &= \sum_{i=0}^{k} V(p^i) \\ &= V(1) + \sum_{i=1}^{k} V(p^i) \\ &= V(1) + \sum_{i=1}^{k} [(p^i - p^{i-1}) + 1] \\ &= 1 + (p + p^2 + ... + p^k) - (1 + p + p^2 + ... + p^{k-1}) + k\end{aligned}$$

$K(p^k) = p^k + k.$ □

Example 4.4.2. *Consider $\mathcal{N} = GR(2^2, 2^2)$, then*

$$\begin{aligned}V(2^2) &= \sum_{t\|} \varphi(t) \\ &= \varphi(1) + \varphi(4) \\ &= 1 + 2 = 3.\end{aligned}$$

Thus the number of regular elements are 3.

Theorem 4.4.3. *Let $\mathcal{N} = GR(p^k, p^k)$ and $\sigma(p^k)$ be the sums of the divisors of p^k.*

Then

$$\begin{aligned}\sigma(p^k) &= \sum_{i=0}^{k} p^i and\\ V(p^k)\sigma(p^k) &= [p^k - p^{k-1}][\sum_{i=0}^{k} p^i].\end{aligned}$$

Proof. Clearly,

$$\begin{aligned}V(p^k)\sigma(p^k) &= [p^k - p^{k-1}][\sum_{i=0}^{k} p^i]\\ &= p^k(1-\frac{1}{p}+\frac{1}{p^k})(\sum_{i=1}^{k} p^i)\\ &= p^k(1-\frac{1}{p}+\frac{1}{p^k})(1+p+p^2+...+p^k)\\ &= p^k[1+p+p^2+...+p^k-\frac{1}{p}-1-p-...p^{k-1}+\frac{1}{p^k}+\frac{1}{p^{k-1}}+\frac{1}{p^2}+\frac{1}{p}+1]\\ &= p^k[1+p^k+p^{-2}+p^{-3}+...+p^{2-k}+p^{1-k}+p^k]\\ &= p^k[1+p^k+\sum_{i=2}^{k} p^{-i}]\\ &= p^{2k}[1+p^{-k}+\sum_{i=2}^{k} p^{-(k+i)}]\end{aligned}$$

which implies that

$$\frac{V(p^k)\sigma(p^k)}{p^{2k}} = 1+p^{-k}+\sum_{i=2}^{k} p^{-(k+i)}$$

as required. □

Theorem 4.4.4. *Let $\mathcal{N} = GR(p^k, p^k)$. Then $\sigma(p^k)+\varphi(p^k) \leq p^k\tau(p^k)$.*

Proof. Let $k = 1$. Then $\sigma(p^k) = p+1$ and $\varphi(p) = p-1$ so that

$\sigma(p)+\varphi(p) = 2p$. Since p has only two divisors 1 and p, this implies that

$2p = p(p\tau)$. Thus $\sigma(p) + \varphi(p) = 2p$. Now suppose that $k > 1$, then,

$$\sigma(p^k) = \sum_{i=1}^{k} p^i$$

and $\varphi(p^k) = p^k - p^{k-1}$, so that

$$\begin{aligned}
\sigma(p^k) + \varphi(p^k) &= 1 + p + ... + p^k + p^k + p^{k-1} \\
&= 2p^k + p^{k-2} + ... + p + 1 < (k+1)p^k.
\end{aligned}$$

But p^k has $(k+1)$ divisors so that $(k+1)p^k = p^k\tau(p^k)$,
thus $\sigma(p^k) + \varphi(p^k) < p^k\tau(p^k)$. □

Example 4.4.3. *Let* $\mathcal{N} = \mathbb{Z}_4[x]/ < x+1 >= GR(2^2, 2^2)$

$$\begin{aligned}
\sigma(2^2) + \varphi(2^2) &\leq 2^2\tau(2^2) \\
\Rightarrow \sigma(4) + \varphi(4) &\leq 4\tau 4 \\
\Rightarrow 7 + 2 &\leq 4 \times 3.
\end{aligned}$$

Thus the result of $\sigma(p^k) + \varphi(p^k) < p^k\tau(p^k)$ holds.

Proposition 4.4.3. *Consider* $\mathcal{N} = GR(p^{kr}, p^k)$*, where* $kr = n > 1$*. Then* $\sigma(p^n) + V(p^n) < p^n\tau(p^n)$*.*

Proof. $1 + \frac{1}{p} + \frac{1}{p^2} + ... + p^n < n = (n+1) - 1 = \tau(p^n) - 1$.

Now

$$\begin{aligned}
\frac{\sigma(p^n)}{p^n} &= \frac{1 + p + p^2 + ... + p^n}{p^n} < \tau(p^n) - 1 \\
\Rightarrow \sigma(p^n) &< \sigma p^n[\tau(p^n) - 1] \\
&= p^n\tau(p^n) - p^n.
\end{aligned}$$

Since $V(p^n) < p^n$, it is clear that $\sigma(p^n) + V(p^n) < p^n\tau(p^n)$. However, if $n = 1$, then $\sigma(p) + V(p) > p\tau(p)$. Let

$$\begin{aligned} \mathcal{N} &= \mathbb{Z}_2[x]/ < x^2 + x + 1 >: p = 2, r = 2, k = 1, n = kr > 1 \\ &= \{\overline{0}, \overline{1}, \overline{x}, \overline{x+1}\}. \end{aligned}$$

We notice that,

$$\begin{aligned} \sigma(p) &= \sigma(2) = 1 + 2 = 3 \\ V(p) &= V(2) = 2 \\ \tau(p) &= \tau(2) = 2 \end{aligned}$$

$$\Rightarrow \sigma(p) + V(p) > p\tau(p) i.e. 5 > 4.$$

But, if $\mathcal{N} = \mathbb{Z}_2[x]/ < x^2 + x + 1 >\cong GR(p^{kr}, p^k), k = 2, r = 2, p = 2$,
$\sigma(p^k) = \sigma(4) = 2$, $V(4) = 4$, $p^k\tau(p^k) = 4\tau(4) = 4 \times 3 = 12$.
Therefore $\sigma(p^k) + V(p^k) < p^k\tau(p^k)(6 < 12)$ which justifies the previous result. □

Lemma 4.4.1. *Let $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \oplus \mathcal{M}$ where p is prime k and r are positive integers and $\mathcal{M}$ is a h-dimensional near-module over $\mathcal{N}$. Then if $h = 0$,*

(i) $R(\mathcal{N}) \cong (1 + Z(\mathcal{N})) \cup \{0\}$ *and*

(ii) $\mid R(\mathcal{N}) \mid= (p^{(k-1)r})(p^r - 1) + 1.$

Proof. Let $a \in R(\mathcal{N}) \cong (1 + Z(\mathcal{N})) \cup \{0\}$. Then a is invertible or 0. But $\mathcal{N}$ is local means that a is regular i.e. $a \in R(\mathcal{N})$.
Thus

$$R(\mathcal{N}) \subseteq [< a > \times(1 + Z(\mathcal{N}))] \cup \{0\} \tag{4.4.6}$$

Conversely, let $a \in R(\mathcal{N})$. Then by definition $\exists$ an element $b \in R(\mathcal{N})$ such that $a = a^2b \Rightarrow a(1 - ab) = 0$.

If $a \in (\mathcal{N}^*)$ then $1 - ab = 0 \Rightarrow ab = 1$.

Hence b is a Von Neumann inverse of a. If a is not a member of $\mathcal{N}^*$ then ab is not a member of $\mathcal{N}^*$, but $ab = aabb = a^2b^2 = abab = (ab)^2$.

Since $\mathcal{N}$ commutes, it implies $ab = (ab)^2 \Rightarrow ab(1 - ab) = 0$.

Now implies $1-ab$ is a unit and $ab = 0$ so that $a = 0$ because b is its Von Neumann inverse.

$$[\{< a > \times 1 + Z(\mathcal{N})\} \cup \{0\}] \subseteq R(\mathcal{N}) \tag{4.4.7}$$

Combining (5.4.1) and (5.4.2) gives

$$\begin{aligned} R(\mathcal{N}) &\cong [1 + Z(\mathcal{N})] \cup \{0\} \\ &= < a > \times [1 + Z(\mathcal{N})] \cup \{0\}. \end{aligned}$$

Next,

$$\begin{aligned} \mathcal{N}^* &= (\mathcal{N}^*/1 + Z(\mathcal{N})) \times 1 + Z(\mathcal{N}) \\ &\cong < a > \times [1 + Z(\mathcal{N})] \\ &= \mathbb{Z}_{p^r - 1} \times [1 + Z(\mathcal{N})]. \end{aligned}$$

But

$$\begin{aligned} \mid [1 + Z(\mathcal{N})] \mid &= \mid Z(\mathcal{N}) \mid \\ &= p^{(k-1)r}. \end{aligned}$$

Therefore $\mid \mathcal{N}^* \mid = (p^r - 1)(p^{(k-1)r})$.

But $R(\mathcal{N}) = \mathcal{N}^* \cup \{0\}$ thus $\mid R(\mathcal{N}) \mid = (p^r - 1)(p^{(k-1)r}) + 1$ as required. $\square$

Theorem 4.4.5. *Let $\mathcal{N}$ be the near-ring constructed in (3.2.1) and (4.2.1) and $R(\mathcal{N})$ be the set of all the regular elements. Then*

(i).

$$R(\mathcal{N}) = \begin{cases} \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h \cup \{0\}, & Char\mathcal{N} = p; \\ \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^{h+1} \cup \{0\}, & Char\mathcal{N} = p^2. \end{cases}$$

(ii).

$$R(\mathcal{N}) = \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}} \times \mathbb{Z}_{2^{k-1}}^{r-1} \times (\mathbb{Z}_2)^h \cup \{0\}, & p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^{k}-1}^r \times (\mathbb{Z}_p^r)^h \cup \{0\}, & p \neq 2 : Char\mathcal{N} = p^k : k \geq 3. \end{cases}$$

Proof. Let $\tau_1, ..., \tau_r \in \mathbb{F}_q$ with $\tau_1 = 1$ such that $\overline{\tau}_1, ..., \overline{\tau}_r$ form a basis for $\mathbb{F}_q$ regarded as a vector space over its prime subnear-field $\mathbb{F}_q$ where $q = p^r$ for any prime p and a positive integer r.

(i) Let $char\mathcal{N} = p$ [case 1]

Observe that for every $l = 1, ..., r$ and $1 \leq i \leq h$, $1 + \tau_l u_i \in 1 + Z(\mathcal{N})$ and

$$\begin{aligned} (1 + \tau_l u_1)^p &= (1 + \tau_l u_1 + \tau_l u_2)^p \\ &= ... \\ &= (1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^p \\ &= 1. \end{aligned}$$

That is, $y^p = 1 \ \forall \ y \in 1 + Z(\mathcal{N})$.

Now for the positive integers $a_{1l}, a_{2l}, ..., a_{hl}$ with $a_{1l} \leq p, a_{2l} \leq p, ..., a_{hl} \leq p$ we notice that

$$\prod_{l=1}^{r}\{(1+\tau_l u_1)^{a_{1l}}\} \cdot \prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2)^{a_{1l}}\} \cdot \\ \dots \cdot \prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2+\dots+\tau_l u_h)^{a_{1l}}\} = 1$$

will imply that $a_{il} = p, \forall l = 1, ..., r$ and $1 \leq i \leq h$.

Let

$$\begin{aligned} S_{1l} &= \{(1+\tau_l u_1)^{a_1} : a_1 = 1, ..., p\} \\ S_{2l} &= \{(1+\tau_l u_1+\tau_l u_2)^{a_2} : a_2 = 1, ..., p\} \\ &\vdots \\ S_{hl} &= \{(1+\tau_l u_1+\dots+\tau_l u_h)^{a_h} : a_h = 1, ..., p\}, \end{aligned}$$

then $S_{1l}, S_{2l}, ..., S_{hl}$ are all cyclic subgroups of $1 + Z(\mathcal{N})$ and they are each of order p, hence $1 + Z(\mathcal{N})$ is abelian and each element $a \in 1 + Z(N)$ is such that $a^p = 1$.

Now

$$\prod_{l=1}^{r} |< 1+\tau_l u_1 >| \cdot \prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2 >| \dots$$

$$\prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2+\dots+\tau_l u_h >| = p^{hr}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the hr subgroups $S_{1l}...S_{hl}$ exhausts $(1 + Z(N))$.

But

$R(\mathcal{N}) =< a > \ltimes (1 + Z(\mathcal{N})) \cup \{0\} : o(a) = p^r - 1 = \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h \cup \{0\}$, which settles the case 1.

Let $p^2 = char\mathcal{N}$ [case 2].

We notice that for every $l = 1, ..., r, (1 + p\tau_l)^p = 1$,
$(1+\tau_l u_1)^{p^2} = 1, (1+\tau_l u_1+\tau_l u_2)^{p^2} = 1, ..., (1+\tau_l u_1+...+\tau_l u_h)^{p^2} = 1)$. For positive integers $a_l, b_{1l}, ..., b_{hl}$ with $a_l \leq p, b_{1l} \leq p^2, ..., b_{hl} \leq p^2$. It is clear that

$$\prod_{l=1}^{r}(1 + p\tau_l)^{a_l}.\prod_{l=1}^{r}(1 + \tau_l u_1)^{b_{1l}}.....$$

$$\prod_{l=1}^{r}(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^{b_{hl}} = 1$$

$\Rightarrow a_l = p, b_{1l} = p^2, ..., b_{hl} = p^2$ for every $l = 1, 2, ..., r$ and $1 \leq i \leq h$.

Set

$$\begin{aligned} T_l &= \{1 + p\tau_l)^a : a = 1, ..., p\} \\ S_{1l} &= \{1 + \tau_l u_1)^{b_1} : b_1 = 1, ..., p^2\} \\ &\vdots \\ S_{hl} &= \{(1 + \tau_l u_1 + \tau_l u_2 +, ..., +\tau_l u_h)^{b_h} : b_h = 1, ..., p^2\}. \end{aligned}$$

$T_l, S_1, ..., S_h$ are all cyclic subgroups of the group $1 + Z(\mathcal{N})$ and they are of the orders indicated by their definitions.

Since
$\prod_{l=1}^{r} |< 1 + p\tau_1 >| ..... \prod_{l=1}^{r} |< 1 + \tau_l u_1 + ... + \tau_l u_h >| = p^{(2h+1)r}$,
and the intersection of any pair of the cyclic subgroups $T_l, ..., S_h$ gives an identity group, the product of the $(h + 1)r$ subgroups $T_l, S_1, ..., S_h$ is direct and exhausts $1 + Z(\mathcal{N})$.

Thus according to case 1, we have $R(\mathcal{N}) =< \alpha > \times(1 + \mathbb{Z}) \cup \{0\}$,
$R(\mathcal{N}) = \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^{h+1} \cup \{0\}$.
(iii) Let char $\mathcal{N} = p^k : k \geq 3$.

We provide the general case using $p =$ odd.

Notice that every

$l = 1, ..., r; (1 + p\tau_1)^{p^{k-1}} = 1$

$(1 + \tau_l u_1)^{p^k} = 1, ..., (1 + p\tau_L u_1 + \tau_l u_2 + ... + \tau_l u_n)^{p^k} = 1.$

Let $a_l, b_{1l}, ..., b_{hl} \in \mathbb{Z}^+$ with $a_l \leq p^{k-1}$,

$b_{il} \leq p^k : 1 \leq i \leq h$. We notice that

$\prod_{l=1}^{r}\{(1 + p\tau_L)^{a_L}\} \cdot \prod_{l=1}^{r}\{(1 + \tau_l u_1)^{b_{1l}}\} \cdot \prod_{l=1}^{r}\{(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)\} = 1$

which $\Rightarrow a_l = p^{k-1}$, $b_{1l} = p^k = \cdots = b_{hl} = p^k$. Set

$$\begin{aligned} T_l &= < \{(1 + p\tau_l)^a \mid a = 1, ..., p^{k-1}\} > \\ S_{1l} &= < \{(1 + \tau_l u_1)^{b_1} \mid b_1 = 1, \cdots, p^k\} > \\ &\vdots \\ S_{hl} &= < \{(1 + \tau_l u_1 + \cdots + \tau_l u_n)^{b_h} \mid b_h = 1, \cdots, p^k\} > . \end{aligned}$$

The sets defined are all cyclic subgroups of the group $1 + Z(\mathcal{N})$ and they are of the indicated orders. Furthermore, the intersection of any pair of the cyclic subgroups indicated gives an identity group and the product of the $(h + 1)r$ subgroups gives: $\mid T_l \times S_{1l} \times S_{hl} \mid = p^{k((h+1)r)-1}$, exhausting $1 + Z(\mathcal{N})$.

Thus $1 + \mathbb{Z}(\mathcal{N}) \cong \mathbb{Z}_{p^{k-1}}^r \times (\mathbb{Z}_p^r)^h$.

Therefore

$$R(\mathcal{N}) =< \alpha > \ltimes (1 + (Z(\mathcal{N})))$$

$$= \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^k-1}^r \times (\mathbb{Z}_p^r)^h.$$

$\square$

Theorem 4.4.6. *Let $\mathcal{N} = R_0 \oplus \mathcal{M}$ where $r = 1$ and p-prime, $k \in \mathbb{Z}^+$.*

If

$$\mathcal{M} = R_0/pR_0 \oplus ... \oplus R_0/pR_0$$

. Let $r_0 \in R(R_0)$ then, its Von-Neumann inverse is
$r_0^{-1} = r_0^{p^k - p^{k-1} - 1}$ and $(r_0, ..., r_h)^{-1} = (r_0^{p^k - p^{k-1} - 1}, -r_1 t_0 r_0^{-1}, ..., -r_h t_0 r_0^{-1})$.

Proof. We know that if $a \in R_0 = G\mathcal{N}(p^{kr}, p^k)$ then, the Von-Neumann inverse of a is given by: $a^{-1} \equiv a^{p^{(k-1)r(p^r-1)-1}} (mod\, p^k)$, therefore

$$r_0^{-1} \equiv r_0^{p^k - p^{k-1} - 1}$$

as required in step 1.

Now let $(t_0, ..., t_h) = (r_0, ..., r_h)^{-1}$, then

$$\begin{aligned}
(r_0, r_1, ..., r_h) &= (r_0, ..., r_h)^2 (t_0, ..., t_h) \\
&= (r_0^2, r_0 r_1 + r_1 r_0, ..., r_0 r_h + r_h r_0)(t_0, ..., t_h) \\
&= (r_0^2 t_0, r_0^2 t_1 + (r_0 r_1 + r_1 r_0) t_0, ..., r_0^2 t_h + (r_0 r_h + r_h r_0) t_0),
\end{aligned}$$

therefore $r_0 = r_0^2 t_0 \Rightarrow r_0 t_0 = 1 \Rightarrow t_0 = r_0^{-1} = r_0^{p^k - p^{k-1} - 1}$.

For $i = 1, ..., h, r_i = r_0^2 t_i + (r_0 r_i + r_i r_0) t_0$

$$\begin{aligned}
\Rightarrow r_0^2 t_i &= r_i - (r_0 r_i + r_i r_0) t_0 \\
\Rightarrow \quad t_i = &\ \frac{r_i - 2 r_0 r_i t_0}{r_0^2} (\therefore \mathcal{N} commutative) \\
\Rightarrow t_i &= \frac{r_i}{r_0^2} - \frac{2 r_i t_0}{r_0}.
\end{aligned}$$

But $t_0 = r_0^{-1}$

$$\begin{aligned}
\Rightarrow t_i &= \frac{r_i}{r_0^2} - \frac{2 r_i}{r_0^2} \\
&= -\frac{r_i}{r_0^2} = -r_i r_0^{-2}.
\end{aligned}$$

$\therefore t_1 = -r_1 r_0^{-2} ... t_h = -r_h r_0^{-2}$

$\Rightarrow (r_0, ..., r_h)^{-1} = (r_0^{p^k - p^{k-1} - 1}, ..., -r_h r_0^{-2})$ as required. □

Example 4.4.4. $\mathcal{N} = \mathbb{Z}_9 \oplus \mathbb{Z}_9/3\mathbb{Z}_9 \oplus ... \oplus \mathbb{Z}_9/3\mathbb{Z}_9$

Then

$$\begin{aligned}(2, \overline{2}, ..., \overline{2})^{-1} &= (2^{9-3-1}, (-2)(5)^2, ..., (-2)(5)^2) \\ &= (5, \overline{1}, \overline{1}, ..., \overline{1})\end{aligned}$$

$(5, \overline{1}, \overline{1}, ..., \overline{1})(2, \overline{2}, ..., \overline{2}) = (1, \overline{0}, ..., \overline{0})$.

Example 4.4.5. *Consider* $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \cong \mathbb{Z}_2[x]/ < x^2 + x + 1 >$, *where* $p = 2, k = 1, r = 2$.

Now $G\mathcal{N} = \{0, 1, x, x + 1\}$ *and* $R(\mathcal{N}) = \{0, 1, x, x + 1\}$.

Let $\mathcal{N} = G\mathcal{N}(4, 2) \oplus G\mathcal{N}(4, 2)$ *with* $G\mathcal{N}(4, 2)$ *as defined above, then:*

$$\mathcal{N} = \{0, 1, x, x + 1\} \oplus \{0, 1, x, x + 1\}$$

$= \{(0, 0), (0, 1), (0, x), (0, x + 1), (1, 0), (1, 1), (1, x), (1, x + 1), (x, 0), (x, 1), (x, x), (x, x + 1), (x + 1, 0), (x + 1, 1), (x + 1, x), (x + 1, x + 1)\}$.

So $| \mathcal{N} |= 16$, $Z_L(\mathcal{N}) = \{(0, 0), (0, 1), (0, x), (0, x + 1)\}$. *Since* $\mathcal{N}$ *is an extension of* $G\mathcal{N}(4, 2)$,

$$| R(N) |= 13 = (p^r - 1)(p^{kr}) + 1.$$

Applying $(r_0, r_1)^{-1} = (r_0^{p^k - p^{k-1} - 1}, -r_1 r_0^{-2})$, *we can find the Von Neumann inverses of all the members of* $R(\mathcal{N})$.

For instance,

$$R(\mathcal{N}) = \{(1, 0), (1, 1), (1, x), (1, x + 1), (x, 0), (x, 1), (x, x), (x, x + 1),$$

$(x + 1, 0), (x + 1, 1), (x + 1, x), (x + 1, x + 1)\}$.

So $(1, 0)^{-1} = (1^{2^1 - 2^0 - 1}, -01^{-1}) = (1^2, 0) = (1, 0)$, $(x, x)^{-1} = (x^{-2}, x^{-1})$.

This can be done in the same manner for the other members of $R(\mathcal{N})$.

The next result gives the structures and orders of the automorphism groups of the regular elements, $R(\mathcal{N})$.

Theorem 4.4.7. *Let $\mathcal{N}$ be a near-ring of construction (3.2.1) and (4.2.1), $R(\mathcal{N})$ be the set of all the regular elements including 0. Then if*

$Aut : R(\mathcal{N}) \to R(\mathcal{N})$ is an automorphism:

(i) when $char\mathcal{N} = p$, then $Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{hr}(G\mathcal{N}(p^r, p))] \cup \triangle$ where $\triangle = \{x \in R(\mathcal{N}) : Aut(x) = 0\}$

(ii) when $char\mathcal{N} = p^2$, then, $Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{(h+1)r}(G\mathcal{N}(p^{2r}, p^2))] \cup \triangle$

(iii) when

$char\mathcal{N} = p^k : k \geq 3$, then,

$Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{(k-1)r}(GN(p^{kr}, p^k))] \times GL_{hr}(GN(p^{kr}, p^k))] \cup \triangle$.

Proof. By enumeration, $R(\mathcal{N}) =< a > \times (1 + Z_L(\mathcal{N})) \cup \{0\}$ where

$< a >= \mathbb{Z}_{p^r-1}$. Since $gcd(p^r - 1, \mid 1 + Z_L(\mathcal{N}) \mid) = 1$, we have that

$Aut(\mathbb{Z}_{p^r-1} \times 1 + Z_L(\mathcal{N})) \cong Aut(\mathbb{Z}_{p^r-1}) \times Aut(1 + Z(\mathcal{N}))$.

But $Aut(\mathbb{Z}_{p^r-1}) = (\mathbb{Z}_{p^r-1})^*$ which is a permutation group whose order coincides with the order of $(1 + Z_L(\mathcal{N}))$.

Next, define a zero automorphism to be the set

$\triangle = \{x \in R(\mathcal{N})\} : Aut(x) = 0$. Then clearly $\triangle = \{Aut(0) = 0_n\}$.

When

$char\mathcal{N} = p$, $R(\mathcal{N}) = [\mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h] \cup \{0\}$

$\Rightarrow AutR(\mathcal{N}) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{hr}(\mathbb{F}_p)] \cup \triangle$ which proves (i). The conditions (ii) and (iii) follow from the proof of (i) with modifications. □

The next result is on the counting of the automorphisms of the regular elements.

Theorem 4.4.8. *Let $\mathcal{N}$ be a zero symmetric local near-rings from the class of near-rings of constructions (3.2.1) and (4.2.1). Then:*

(i)

$$| Aut(R(\mathcal{N})) | = [\varphi(p^r - 1) \cdot \prod_{k=1}^{hr}(p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p$

(ii)

$$| Aut(R(\mathcal{N})) | = [\varphi(p^r - 1) \cdot \prod_{k=1}^{(h+1)r}(p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p^2$

(iii)

$$| Aut(R(\mathcal{N})) | = [\varphi(p^r - 1) \cdot \prod_{k=1}^{(k-1)r}(p^k - p^{k-1}) \cdot \prod_{k=1}^{hr}(p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p^k : k \geq 3$.

Proof. (i) Let $char\mathcal{N} = p$.
By definition of $\varphi(n)$ attributed to Osama and Emad [55],

$$| (\mathbb{Z}_{p^r-1})^* | = \varphi(p^r - 1)$$

and since $| Aut(\mathbb{Z}_{p^r-1}) | = | (\mathbb{Z}_{p^r-1})^* | = \varphi(p^r - 1)$ the prefix of right hand side to (i) is clear.
From the previous theorem, $Aut(1 + Z_L(\mathcal{N})) = GL_{hr}(\mathbb{Z}_p)$. Thus, we need to find all the elements of

$$GL_{hr}(\mathbb{Z}_p)$$

in the endomorphism, $End(1+Z_L(\mathcal{N}))$ and calculate the distinct ways of extending such an element to an endomorphism. So we need all such matrices that are invertible modulo p.

Let $R_p \in End(1+Z_L(\mathcal{N}))$, then the number of matrices $A \in R_p$ that are invertible modulo p are upper block triangular matrices whose number can be given as:

$$\sharp A = \prod_{k=1}^{n}(p^k - p^{k-1}).$$

Now when $char R = p$, $n = hr$ therefore

$$\sharp A = \prod_{k=1}^{hr}[p^k - p^{k-1}].$$

This means that

$$\mid \mathbb{Z}_{p^r-1} \times GL_{hr}(\mathbb{Z}_p)) \mid = \varphi(p^r - 1) \centerdot \prod_{k=1}^{hr}[p^k - p^{k-1}].$$

Finally $0 \in R(\mathcal{N})$ and $Aut(0) = 0$. Now $\mid Aut(0) \mid=\mid \{0\} \mid= 1$, thus

$$\begin{aligned} \mid AutR(\mathcal{N}) \mid &= \mid [Aut(Z_{p^r-1}) \centerdot Aut(GL_{hr}(\mathbb{Z}_p))] \mid + \mid Aut\{0\} \mid \\ &= [\varphi(p^r - 1) \centerdot \prod_{k=1}^{hr}(p^k - p^{k-1})] + 1 \end{aligned}$$

as required. The proofs to (ii) and (iii) are similar to proofs of (i) with modifications on the orders of $GL_n(\mathbb{Z}_p)$. $\square$

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

This study was set up with an aim of determining and classifying the regular elements and Von-Neumann inverses of the zero symmetric local near-rings with n-nilpotent radical of Jordan ideals admitting Frobenius derivations. The objectives have been achieved via three main steps that is: determination of constructions (3.2.1) and (4.2.1) representing the classes of the near-rings under investigations whose algebraic structures were subjected to commutation checks using the theorem of Asma and Inzamam involving properties of $\mathcal{J}(\mathcal{N})$ and the Frobenius derivations, $d : \mathcal{N} \to \mathcal{N}$ and $d : \mathcal{N} \to \mathcal{J}(\mathcal{N})$.

The structures and orders of $R(\mathcal{N})$ were then characterized in a case by case basis using the fundamental theorem of Finitely Generated Abelian Groups and the properties of the general linear groups in the endomorphism of $R(\mathcal{N})$ respectively. The structures of $V(\mid R(\mathcal{N}) \mid)$ followed asymptotic patterns proposed by Osama and Emad [55] using the properties of $V(n)$, $\tau(n)$, $\overline{\omega}(n)$, $\sigma(n)$ and $K(n)$. The morphisms of $\Gamma(\mathcal{N})$ followed number theoretic analysis and the automorphism of $\Gamma(\mathcal{N})$ revealed in terms of groups of symmetry.

In conclusion, the research demonstrates that the classes of the near-rings constructed have algebraic properties related to the dual properties of the idealized rings. The Frobenius derivations assigned to the permutation of the products is an automorphism. Thus, this study extends the ring theoretic notion of unit groups to a characterization of $R(\mathcal{N})$. The results of the study regarding the characterization of $\mathcal{N}$, $\Gamma(\mathcal{N})$, $d : \mathcal{N} \to \mathcal{N}$, $d : \mathcal{N} \to \mathcal{J}\mathcal{N})$ and commutation have been presented in theorem 3.2.1, proposition 3.2.1, theorem 3.2.2, theorem 3.3.1 and theorem 3.3.3. The results of propositions 3.4.1 to 3.4.6 show the structures and orders of automorphism of $\Gamma(\mathcal{N})$. Similarly, the properties of $R(\mathcal{N})$, $V(\mid R(\mathcal{N}) \mid)$ and the automorphisms of $R(\mathcal{N})$ are captured in theorems 4.3.5, 4.3.6 and 4.3.8.

5.2 Recommendations

The classification problem of finite rings using near-rings is still open. This study leaves certain gaps that can be considered in future:

(i). A characterization of the regular elements and inverses of non-local near-rings, $\mathcal{N}$ with generalized derivations.

(ii). A determination of the automorphisms of $\mathcal{N}$, unit groups and the generalized graphs of the near-rings in constructions (3.2.1) and (4.2.1).

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